3.6. Uniform Boundedness Principle

Note. This section deals with showing that a subset $\mathcal{A} \subseteq \mathcal{B}(X,Y)$ is bounded. Notice that set $\mathcal{A}$ is a set of bounded linear operators. Just because each element of $\mathcal{A}$ is bounded, that does not mean that set $\mathcal{A}$ itself is necessarily bounded (consider $\mathbb{N}$). As the text mentions, boundedness of set $\mathcal{A}$ is often called “uniform boundedness.”

Definition. Set $\mathcal{A} \subseteq \mathcal{B}(X,Y)$ is pointwise bounded if for each $x \in X$ the set $O_x = \{ T x \mid T \in \mathcal{A} \}$ is bounded in $Y$.

Note. We can think of $O_x$ as sort of an $x$-cross section of set $\mathcal{A}$. We might think of $TX = \{ T x \mid x \in X \}$ as a sort of $y$-cross section, although it is $T$ that is constant in $\mathcal{A}$ and not $y \in Y$ that is held constant. With this interpretation, the following result simply says that if all $x$-cross sections are bounded and all $y$-cross sections are bounded (or $T$-cross sections if you like—this follows, of course, from the fact that each $T$ if bounded) then set $\mathcal{A}$ is bounded.

Theorem 3.10. Uniform Boundedness Principle.

If $X$ is complete, then a pointwise bounded subset $\mathcal{A}$ of $\mathcal{B}(X,Y)$ is bounded.
Note. Sometimes the Uniform Boundedness Principle is called the Banach-Steinhaus Theorem. The following application of the Uniform Boundedness Principle shows that $\mathcal{B}(X,Y)$ is closed under pointwise limits.

**Theorem 3.11.** Suppose that $(T_n)$ is a pointwise convergent sequence of bounded linear operators from Banach space $X$ to normed linear space $Y$. That is, for each $x \in X$ the sequence $(T_n x)$ converges to an element $Tx \in Y$. Then $T$ is linear and bounded.

*Revised: 5/20/2015*