9.5. Compact Self Adjoint Operators on Hilbert Spaces

Note. In this section we give a spectral theorem for a compact self adjoint operator on a Hilbert space.

Note. Of course $\mathbb{R}^n$ and $\mathbb{C}^n$ are Hilbert spaces. We know that every linear transformation $T$ from $\mathbb{R}^n$ to $\mathbb{R}^n$ (or $\mathbb{C}^n$ to $\mathbb{C}^n$) is represented by an $n \times n$ matrix, $A_T$ (see my class notes for Linear Algebra [MATH 2010], Section 2.3, the “Standard Matrix Representation of Linear Transformations”: http://faculty.etsu.edu/gardnerr/2010/c2s3.pdf). A real symmetric matrix is (real) diagonalizable if it is symmetric (see the “Fundamental Theorem of Real Symmetric Matrices”: http://faculty.etsu.edu/gardnerr/2010/c6s3.pdf). A complex matrix is diagonalizable if it is conjugate symmetric (that is, $a_{ij} = \overline{a_{ji}}$). As observed in Section 4.6 (see the class notes for this section, page 3) these matrices correspond to self adjoint operators on finite dimensional spaces. We now turn to infinite dimensional spaces.

Definition. A subspace $M$ of a linear space is *invariant* under linear operator $T$ if $TM \subset M$.

Proposition 9.17. If $M$ is invariant for compact, self adjoint operator $T$ on a Hilbert space then $M^\perp$ is invariant for $T$. Moreover, the restrictions of $T$ to both $M$ and $M^\perp$ are also self adjoint.
Let $T$ be a compact, self adjoint operator on a Hilbert space $H$. There is a sequence (either finite or countably infinite) of mutually orthogonal closed subspaces $(M_n)$ whose closed linear span is all of $H$. There is a corresponding sequence $(\lambda_n)$ of real numbers which if countably infinite converges to 0. For all $n$ and $x \in M_n$, we have $Tx = \lambda_n x$. Moreover, if $\lambda_n \neq 0$ then $M_n$ is finite dimensional.

Note. Theorem 9.18 also holds for normal operators (though the eigenvalues may not be real). This is to be proved in Exercise 9.11.

Theorem 9.19. For $T$ a compact, self adjoint operator on Hilbert space $H$, $T = \sum_n \lambda_n E_{\lambda_n}$ in which $E_{\lambda_n}$ is the projection onto $M_n$ where $M_n$ is the eigenspace associated with $\lambda_n$.

Note. Suppose $H$ is an infinite dimensional separable Hilbert space and let $T$ be a compact, self adjoint operator on $H$. Then there is a sequence $(\lambda_n)$ of eigenvalues of $T$ and eigenspaces $M_n$ such that $H$ is the closed linear span of the $M_n$’s by Theorem 9.18. Let $B_n$ be an orthonormal basis for eigenspace $M_n$ (so each $B_n$ is finite, unless 0 is an eigenvalue in which case the eigenspace $N(T)$ need not be finite dimensional according to Theorem 9.18). Take the union of all these bases, say $B = \{e_k \mid k \in \mathbb{N}\} = \cup_n B_n$. Let $\mu_k$ be the eigenvalue corresponding to the eigenvector $e_k$ (this yields sequence $(\mu_k)_{k=1}^{\infty}$ with $\mu_k$’s repeated according to the dimension of the corresponding eigenspace). We know from Theorem 9.19 that
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\[ T = \sum_k \lambda_k E_{\lambda_k}; \] here we have \( \lambda_k = \mu_k \) and \( E_{\lambda_k}(x) = \langle x, e_k \rangle e_k \) (technically, in Theorem 9.19, \( E_{\lambda_k} \) is the projection onto \( M_k \) and we need to sum over all \( e_k \) in the basis for \( M_k \) to get such an \( E_{\lambda_k} \), but we ultimately take such a sum, as follows).

So \[ T x = \sum_k \mu_k \langle x, e_k \rangle e_k. \]

**Definition.** An operator \( S \) on a Hilbert space \( K \) (so \( S : K \to K \)) is **unilaterily equivalent** to an operator \( T \) on Hilbert space \( H \) (so \( T : H \to H \)) if there is a bijective isometry \( U : K \to H \) such that \( S = U^{-1}TU \).

**Note.** We have the mappings:

\[ K \xrightarrow{U} H \]
\[ S \quad \quad \quad \quad \quad \quad T \]
\[ K \xleftarrow{U^{-1}} H \]

**Theorem 9.20.** A compact, self adjoint operator \( T \) on a separable Hilbert space is unitarily equivalent to a multiplication operator \( M_f \) on \( \ell^2 \).

**Note.** Since \( \ell^2 \) is a relatively conceptually tangible Hilbert space and multiplication operators are also tangible, then Theorem 9.20 gives a nice way to think about compact, self adjoint operators (on separable Hilbert spaces).

*Revised: 5/21/2017*