Introduction to Special Relativity

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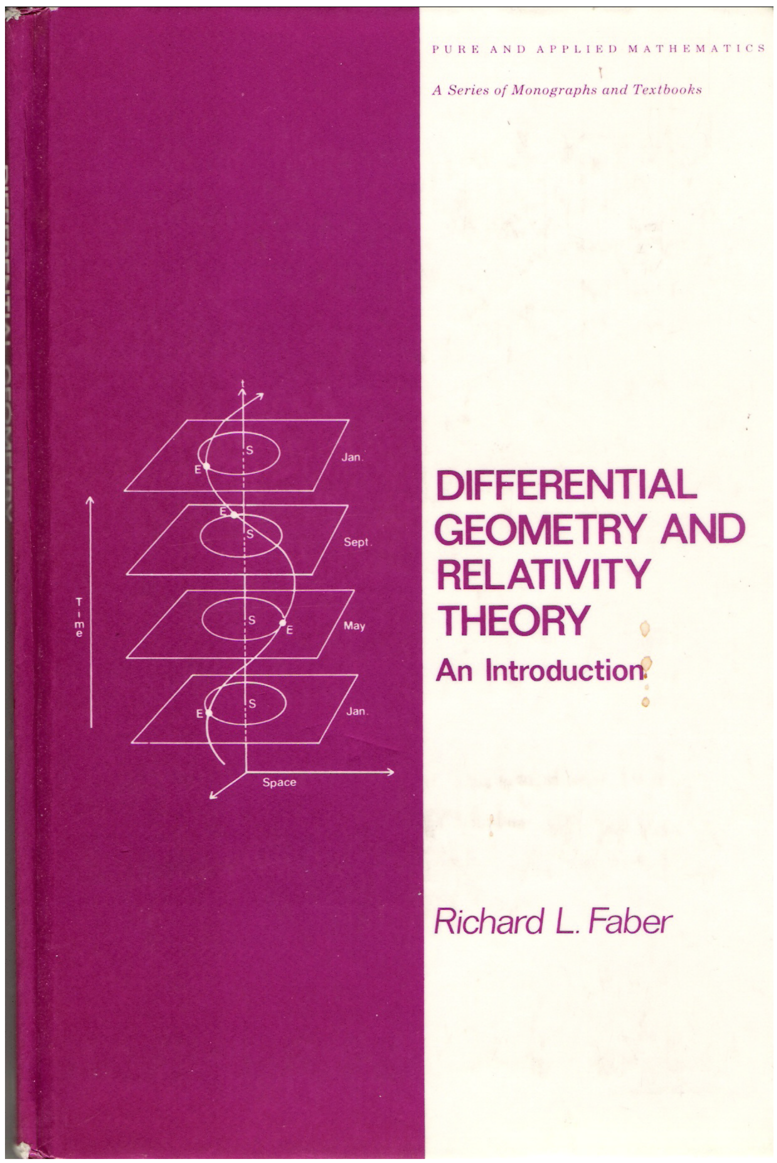
Great Ideas in Science (PHYS 2018)

Notes based on *Differential Geometry and Relativity Theory: An Introduction* by Richard Faber



Chapter 2. Special Relativity: The Geometry of Flat Spacetime

Note. Classically (i.e in Newtonian mechanics), *space* is thought of as

1. unbounded and infinite,

2. 3-dimensional and explained by Euclidean geometry, and

3. "always similar and immovable" (Newton, *Principia Mathematica*, 1687).

This would imply that one could set up a system of spatial coordinates (x, y, z) and describe any dynamical event in terms of these spatial coordinates and time t.

Note. Newton's Three Laws of Motion:

1. (*The Law of Inertia*) A body at rest remains at rest and a body in motion remains in motion with a constant speed and in a straight line, unless acted upon by an outside force.

2. The acceleration of an object is proportional to the force acting upon it and is directed in the direction of the force. That is, $\vec{F} = m\vec{a}$.

3. To every action there is an equal and opposite reaction.

Note. Newton also stated his Law of Universal Gravitation in *Principia*:

> "Every particle in the universe attracts every other particle in such a way that the force between the two is directed along the line between them and has a magnitude proportional to the product of their masses and inversely proportional to the square of the distance between them."

Symbolically, $F = \dfrac{GMm}{r^2}$ where F is the magnitude of the force, r the distance between the two bodies, M and m are the masses of the bodies involved and G is the *gravitational constant* $(6.67 \times 10^{-8}$ cm./(g sec^2)). Assuming only Newton's Law of Universal Gravitation and Newton's Second Law of Motion, one can derive Kepler's Laws of Planetary Motion.

2.1 Inertial Frames of Reference

Definition. A *frame of reference* is a system of spatial coordinates and possibly a temporal coordinate. A frame of reference in which the Law of Inertia holds is an *inertial frame* or *inertial system.* An observer at rest (i.e. with zero velocity) in such a system is an *inertial observer.*

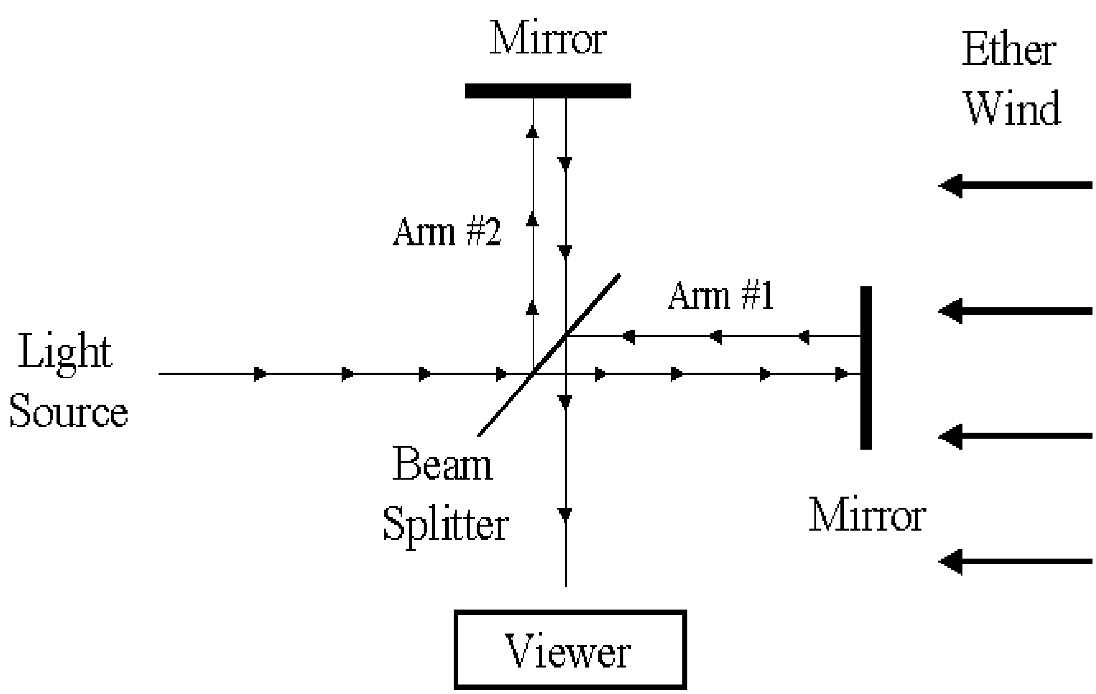
Note. The main idea of an inertial observer in an inertial frame is that the observer experiences no acceleration (and therefore no net force). If S is an inertial frame and S' is a frame (i.e. coordinate system) moving uniformly relative to S, then S' is itself an inertial frame. Frames S and S' are equivalent in the sense that there is no mechanical experiment that can be conducted to determine whether either frame is at rest or in uniform motion (that is, there is **no preferred frame**). This is called the *Galilean* (or *classical*) *Principle of Relativity.*

Note. Special relativity deals with the observations of phenomena by inertial observers and with the comparison of observations of inertial observers in equivalent frames (i.e. NO ACCELERATION!). General relativity takes into consideration the effects of acceleration (and therefore gravitation) on observations.

2.2 The Michelson-Morley Experiment

Note. Sound waves need a medium though which to travel. In 1864 James Clerk Maxwell showed that light is an electromagnetic wave. Therefore it was assumed that there is an *ether* which propagates light waves. This ether was assumed to be everywhere and unaffected by matter. This ether could be used to determine an absolute reference frame (with the help of observing how light propagates through the ether).

Note. The Michelson-Morley experiment (circa 1885) was performed to detect the Earth's motion through the ether as follows:



. The viewer will see the two beams of light which have traveled along different arms display some interference pattern. If the system is rotated, then the influence of the "ether wind" *should* change the time the beams of light take to travel along the arms and therefore should change the interference pattern. The experiment was performed at different times of the day and of the year. NO CHANGE IN THE INTERFERENCE PATTERN WAS OBSERVED!

Note. In 1892, Fitzgerald proposed that an object moving through the ether wind with velocity v experiences a contraction in the direction of the ether wind of $\sqrt{1 - v^2/c^2}$. That is, in the diagram above, L_1 is contracted to $L_1\sqrt{1 - v^2/c^2}$ and then we get $t_1 = t_2$ when $L_1 = L_2$, potentially explaining the results of the Michelson-Morley experiment. This is called the Lorentz-Fitzgerald contraction. Even under this assumption, "it turns out" that the Michelson-Morley apparatus with unequal arms will exhibit a pattern shift over a 6 month period as the Earth changes direction in its orbit around the Sun. In 1932, Kennedy and Thorndike performed such an experiment and detected no such shift.

Conclusion. The speed of light is constant and the same in all directions and in all inertial frames.

2.3 The Postulates of Relativity

Note. Albert Einstein published "*Zur Elektrodynamik bewegter Körper*" (On the Electrodynamics of Moving Bodies) in *Annalen der Physik* (Annals of Physics) **17** (1905). In this paper, he established the **SPECIAL THEORY OF RELATIVITY**! I quote (from "The Principles of Relativity" by H. A. Lorenz, A. Einstein, H. Minkowski, and H. Weyl, published by Dover Publications):

> "...the same laws of electrodynamics and optics will be valid for all frames of reference for which the equations of mechanics hold good. We raise this conjecture (the purport of which will hereafter be called the "Principle of Relativity") to the status of a postulate, and also introduce another postulate, which is only apparently irreconcilable with the former, namely, that light is always propagated in empty space with a definite velocity c which is independent of the state of motion of the emitting body."

In short:

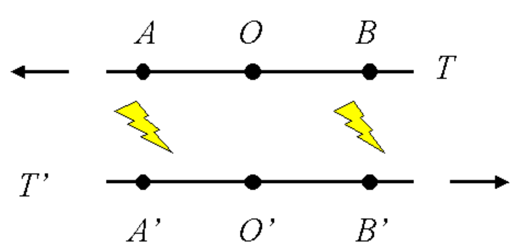
P1. All physical laws valid in one frame of reference are equally valid in any other frame moving uniformly relative to the first.

P2. The speed of light (in a vacuum) is the same in all inertial frames of reference, regardless of the motion of the light source.

From these two simple (and empirically varified) assumptions arises the beginning of the revolution that marks our transition from classical to modern physics!

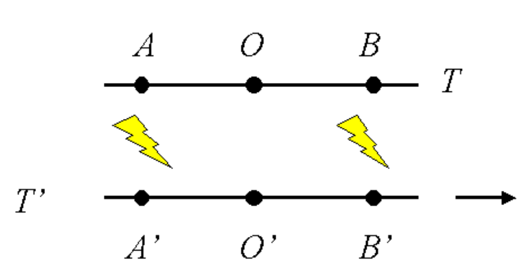
2.4 Relativity of Simultaneity

Note. Suppose two trains T and T' pass each other traveling in opposite directions (this is equivalent to two inertial frames moving uniformly relative to one another). Also suppose there is a flash of lightening (an emission of light) at a certain point. Mark the points on trains T and T' where this flash occurs at A and A' respectively. "Next," suppose there is another flash of lightning and mark the points B and B'. Suppose point O on train T is midway between points A and B, AND that point O' on train T' is midway between points A' and B'. An outsider *might* see:

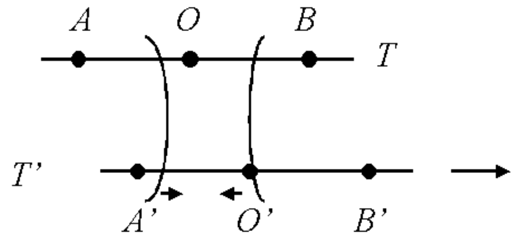


Suppose an observer at point O sees the flashes at points A and B occur at the same time. From the point of view of O the sequence of events is:

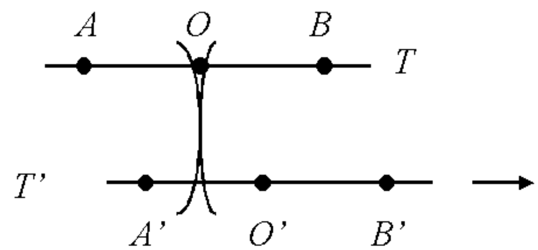
(1) Both flashes occur, A, O, B opposite A', O', B', resp.



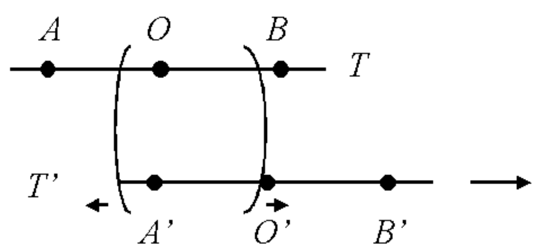
(2) Wavefront from BB' meets O'



(3) Both wavefronts meet O

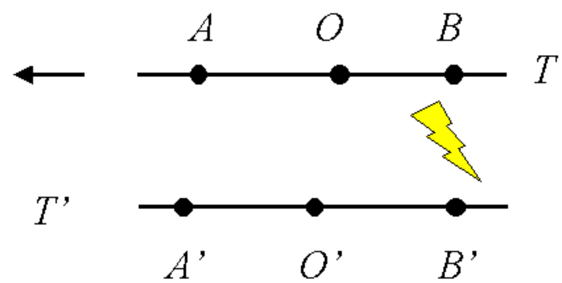


(4) Wavefront from AA' meets O'

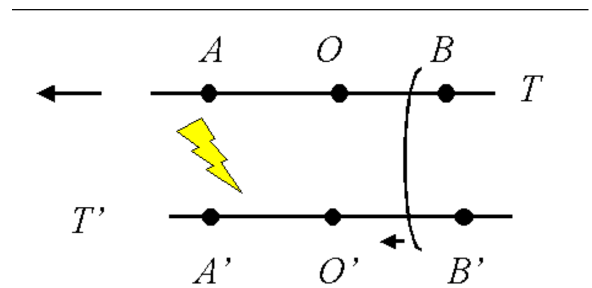


From the point of view of an observer at O', the following sequence of events are observed:

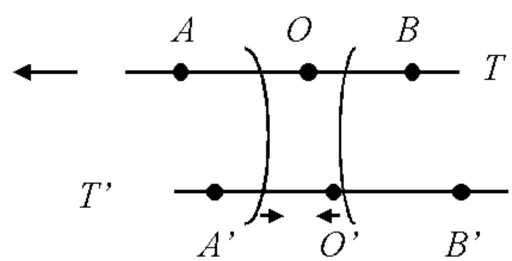
(1) Flash occurs at BB'



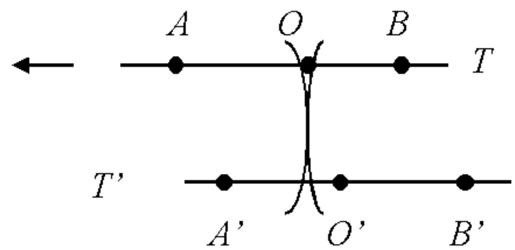
(2) Flash occurs at AA'



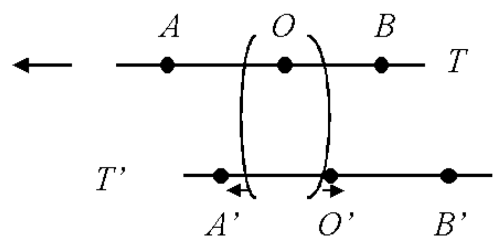
(3) Wavefront from BB' meets O'



(4) Wavefronts from AA' and BB' meet O



(5) Wavefront from AA' meets O'



Notice that the speed of light is the same in both frames of reference. However, the observer on train T sees the flashes occur simultaneously, whereas the observer on train T' sees the flash at BB' occur before the flash at AA'. Therefore, events that appear to be simultaneous in one frame of reference, may not appear to be simultaneous in another. This is the *relativity of simultaneity*.

Note. The relativity of simultaneity has implications for the measurements of lengths. In order to measure the length of an object, we must measure the position of both ends of the object *simultaneously.* Therefore, if the object is moving relative to us, there is a problem. In the above example, observer O sees distances AB and $A'B'$ equal, but observer O' sees AB shorter than $A'B'$. Therefore, we see that measurements of lengths are *relative*!

2.5 Coordinates

Definition. In 3-dimensional geometry, positions are represented by points (x, y, z). In physics, we are interested in *events* which have both time and position (t, x, y, z). The collection of all possible events is *spacetime*.

Definition. With an event (t, x, y, z) in spacetime we associate the units of cm with coordinates x, y, z. In addition, we express t (time) in terms of cm by multiplying it by c. (In fact, many texts use coordinates (ct, x, y, z) for events.) These common units (cm for us) are called *geometric units*.

Note. We express velocities in dimensionless units by dividing them by c. So for velocity v (in cm/sec, say) we associate the dimensionless velocity $\beta = v/c$. Notice that under this convention, the speed of light is 1.

Note. In an inertial frame S, we can imagine a grid laid out with a clock at each point of the grid. The clocks can by synchronized (see page 118 for details). When we mention that an object is *observed* in frame S, we mean that all of its parts are measured simultaneously (using the synchronized clocks). This can be quite different from what an observer at a point actually *sees*.

Note. From now on, when we consider two inertial frames S and S' moving uniformly relative to each other, we adopt the conventions:

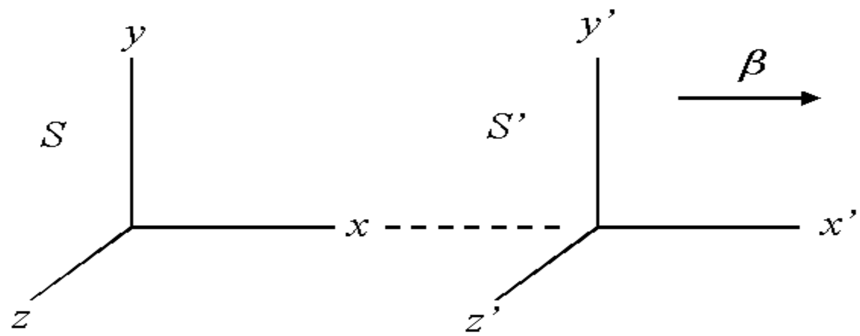
1. The $x-$ and $x'-$axes (and their positive directions) coincide.

2. Relative to S, S' is moving in the positive x direction with velocity β.

3. The $y-$ and $y'-$axes are always parallel.

4. The $z-$ and $z'-$axes are always parallel.

We call S the *laboratory frame* and S' the *rocket frame*:



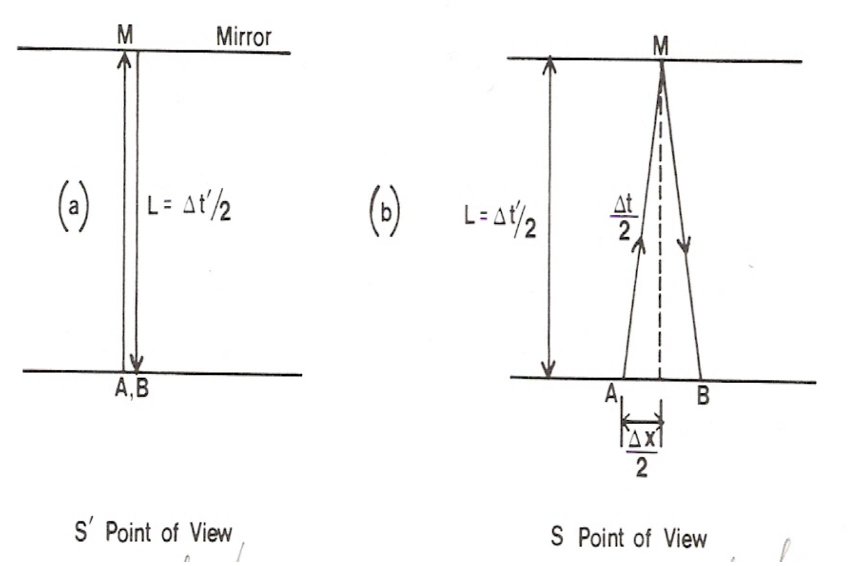
Assumptions. We assume space is *homogeneous* and *isotropic*, that is, space appears the same at all points (on a sufficiently large scale) and appears the same in all directions.

Note. In the next section, we'll see that things are much different in the direction of motion.

2.6 Invariance of the Interval

Note. In this section, we define a quantity called the "interval" between two events which is invariant under a change of spacetime coordinates from one inertial frame to another (analogous to "distance" in geometry). We will also derive equations for time and length dilation.

Note. Consider the experiment described in Figure II-8:



In inertial frame S' a beam of light is emitted from the origin, travels a distance L, hits a mirror and returns to the origin. If $\Delta t'$ is the amount of time it takes the light to return to the origin, then $L = \Delta t'/2$ (recall that t' is multiplied by c in order to put it in geometric units). An observer in frame S sees the light follow the path of Figure II-8b in time Δt. Notice that the situation here is not symmetric since the laboratory observer requires two clocks (at two positions) to determine Δt, whereas the rocket observer only needs one clock (so the Principle of Relativity does not apply). In geometric units, we have: $(\Delta t/2)^2 = (\Delta t'/2)^2 + (\Delta x/2)^2$ or $(\Delta t')^2 = (\Delta t)^2 - (\Delta x)^2$ with β the velocity of S' relative to S, we have $\beta = \Delta x/\Delta t$ and so $\Delta x = \beta \Delta t$ and $(\Delta t')^2 = (\Delta t)^2 - (\beta \Delta t)^2$ or

$$\Delta t' = \sqrt{1-\beta^2}\Delta t. \tag{78}$$

Therefore we see that under the hypotheses of relativity, time is not absolute and the time between events depends on an observer's motion relative to the events.

Note. You might be more familiar with equation (78) in the form:

$$\Delta t = \frac{1}{\sqrt{1-\beta^2}}\Delta t'$$

where $\Delta t'$ is an interval of time in the rocket frame and Δt is how the laboratory frame measures this time interval. Notice $\Delta t \geq \Delta t'$ so that time is dilated (lengthened).

Note. Since $\beta = v/c$, for $v \ll c$, $\beta \approx 0$ and $\Delta t' \approx \Delta t$.

Definition. Suppose events A and B occur in inertial frame S at (t_1, x_1, y_1, z_1) and (t_2, x_2, y_2, z_2), respectively, where $y_1 = y_2$ and $z_1 = z_2$. Then define the *interval* (or *proper time*) between A and B as $\Delta\tau = \sqrt{(\Delta t)^2 - (\Delta x)^2}$ where $\Delta t = t_2 - t_1$ and $\Delta x = x_2 - x_1$.

Note. As shown above, in the S' frame

$$(\Delta t')^2 - (\Delta x')^2 = (\Delta t)^2 - (\Delta x)^2$$

(recall $\Delta x' = 0$). So $\Delta\tau$ is the same in S'. That is, the interval is invariant from S to S'. As the text says "The interval is to spacetime geometry what the distance is to Euclidean geometry."

Note. We could extend the definition of interval to motion more complicated than motion along the $x-$axis as follows:

$$\Delta\tau = \{(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2\}^{1/2}$$

or

$$(\text{interval})^2 = (\text{time separation})^2 - (\text{space separation})^2.$$

Note. Let's explore this "time dilation" in more detail. In our example, we have events A and B occuring in the S' frame at the same position ($\Delta x' = 0$), but at different times. Suppose for example that events A and B are separated by one time unit in the S' frame ($\Delta t' = 1$). We could then represent the ticking of a second hand on a watch which is stationary in the S' frame by these two events. An observer in the S frame then measures this $\Delta t' = 1$ as

$$\Delta t = \frac{1}{\sqrt{1-\beta^2}}\Delta t'.$$

That is, an observer in the S frame sees the one time unit *stretched* (dilated) to a length of $\dfrac{1}{\sqrt{1-\beta^2}} \geq 1$ time unit. So the factor $\dfrac{1}{\sqrt{1-\beta^2}}$ shows how much slower a moving clock ticks in comparison to a stationary clock. The Principle of Relativity implies that on observer in frame S' will see a clock stationary in the S frame tick slowly as well. **However**, the Principle of Relativity does not apply in our example above (see p. 123) and both an observer in S and an observer in S' agree that Δt and $\Delta t'$ are related by

$$\Delta t = \frac{1}{\sqrt{1-\beta^2}}\Delta t'.$$

So **both** agree that $\Delta t \geq \Delta t'$ *in this case.* This seems strange initially, but will make more sense when we explore the *interval* below. (Remember, $\Delta x \neq 0$.)

Definition. An interval in which time separation dominates and $(\Delta\tau)^2 > 0$ is *timelike.* An interval in which space separation dominates and $(\Delta\tau)^2 < 0$ is *spacelike.* An interval for which $\Delta\tau = 0$ is *lightlike.*

Note. If it is possible for a material particle to be present at two events, then the events are separated by a timelike interval. No material object can be present at two events which are separated by a spacelike interval (the particle would have to go faster than light). If a ray of light can travel between two events then the events are separated by an interval which is lightlike. We see this in more detail when we look at spacetime diagrams (Section 2.8).

Note. If an observer in frame S' passes a "platform" (all the train talk is due to Einstein's original work) of length L in frame S at a speed of β, then a laboratory observer on the platform sees the rocket observer pass the platform in a time $\Delta t = L/\beta$. As argued above, the rocket observer measures this time period as $\Delta t' = \Delta t\sqrt{1-\beta^2}$. Therefore, the rocket observer sees the platform go by in time $\Delta t'$ and so measures the length of the platform as

$$L' = \beta\Delta t' = \beta\Delta t\sqrt{1-\beta^2} = L\sqrt{1-\beta^2}.$$

Therefore we see that the time dilation also implies a length contraction:

$$L' = L\sqrt{1-\beta^2}. \tag{83}$$

Note. Equation (83) implies that lengths are contracted when an object is moving fast relative to the observer. Notice that with $\beta \approx 0$, $L' \approx L$.

Example (Exercise 2.6.2). Pions are subatomic particles which decay radioactively. At rest, they have a half-life of 1.8×10^{-8} sec. A pion beam is accelerated to $\beta = 0.99$. According to classical physics, this beam should drop to one-half its original intensity after traveling for $(0.99)(3 \times 10^8)(1.8 \times 10^{-8}) \approx 5.3$m. However, it is found that it drops to about one-half intensity after traveling 38m. Explain, using either time dilation or length contraction.

Solution. Time is not absolute and a given amount of time $\Delta t'$ in one inertial frame (the pion's frame, say) is observed to be dilated in another inertial frame (the particle accelerator's) to $\Delta t = \Delta t'/\sqrt{1-\beta^2}$. So with $\Delta t' = 1.8 \times 10^{-8}$sec and $\beta = 0.99$,

$$\Delta t = \frac{1}{\sqrt{1 - .99^2}}(1.8 \times 10^{-8}\text{sec}) = 1.28 \times 10^{-7}\text{sec}.$$

Now with $\beta = .99$, the speed of the pion is $(.99)(3 \times 10^8\text{m/sec}) = 2.97 \times 10^8\text{m/sec}$ and in the inertial frame of the accelerator the pion travels

$$(2.97 \times 10^8\text{m/sec})(1.28 \times 10^{-7}\text{sec}) = 38\text{m}.$$

In terms of length contraction, the accelerator's length of 38m is contracted to a length of

$$L' = L\sqrt{1-\beta^2} = (38\text{m})\sqrt{1 - .99^2} = 5.3\text{m}$$

in the pion's frame. With $v = .99c$, the pion travels this distance in

$$\frac{5.3\text{m}}{(.99)(3 \times 10^8\text{m/sec})} = 1.8 \times 10^{-8}\text{sec}.$$

This is the half-life and therefore the pion drops to 1/2 its intensity after traveling 38m in the accelerator's frame.

2.7 The Lorentz Transformation

Note. We seek to find the transformation of the coordinates (x, y, z, t) in an inertial frame S to the coordinates (x', y', z', t') in inertial frame S'. Throughout this section, we assume the x and x' axes coincide, S' moves with velocity β in the direction of the positive x axis, and the origins of the systems coincide at $t = t' = 0$. See Figure II-9, page 128.

Note. Classically, we have the relations

$$\begin{aligned} x &= x' + \beta t \\ y &= y' \\ z &= z' \\ t &= t' \end{aligned}$$

Definition. The assumption of *homogeneity* says that there is no preferred location in space (that is, space looks the same at all points [on a sufficiently large scale]). The assumption of *isotropy* says that there is no preferred direction in space (that is, space looks the same in every direction).

Note. Under the assumptions of homogeneity and isotropy, the relations between (x, y, z, t) and $(x'.y', z', t')$ must be linear (throughout, everything is done in geometric units!):

$$\begin{aligned} x &= a_{11}x' + a_{12}y' + a_{13}z' + a_{14}t' \\ y &= a_{21}x' + a_{22}y' + a_{23}z' + a_{24}t' \\ z &= a_{31}x' + a_{32}y' + a_{33}z' + a_{34}t' \\ t &= a_{41}x' + a_{42}y' + a_{43}z' + a_{44}t'. \end{aligned}$$

If not, say $y = ax'^2$, then a rod lying along the $x-$axis of length $x_b - x_a$ would get longer as we moved it out the $x-$axis, contradicting homogeneity. Similarly, relationships involving time must be linear (since the length of a time interval should not depend on time itself, nor should the length of a spatial interval).

Note. We saw in Section 2.5 that lengths perpendicular to the direction of motion are invariant. Therefore

$$\begin{aligned} y &= y' \\ z &= z' \end{aligned}$$

Note. x does not depend on y' and z'. Therefore, the coefficients a_{12} and a_{13} are 0. Similarly, isotropy implies $a_{42} = a_{43} = 0$. We have reduced the system of equations to

$$x = a_{11}x' + a_{14}t' \tag{85}$$

$$t = a_{41}x' + a_{44}t' \tag{86}$$

Definition. The transformation relating coordinates (x, y, z, t) in S to coordinates (x', y', z', t') in S' given by

$$\begin{aligned} x &= \frac{x' + \beta t'}{\sqrt{1-\beta^2}} \\ y &= y' \\ z &= z' \\ t &= \frac{\beta x' + t'}{\sqrt{1-\beta^2}} \end{aligned}$$

is called the *Lorentz Transformation.*

Note. With $\beta \ll 1$ and $\beta^2 \approx 0$ we have

$$\begin{aligned} x &= x' + \beta t' \\ t &= t' \end{aligned}$$

(in geometric units, x and x' are small compared to t and t' and $\beta x'$ is negligible compared to t', but $\beta t'$ is NOT negligible compared to x').

Note. By the Principle of Relativity, we can invert the Lorentz Transformation simply by interchanging x and t with x' and t', respectively, and replacing β with $-\beta$!

Note. If we deal with pairs of events separated in space and time, we denote the differences in coordinates with Δ's to get

$$\Delta x = \frac{\Delta x' + \beta \Delta t'}{\sqrt{1-\beta^2}} \qquad (91a)$$

$$\Delta t = \frac{\beta \Delta x' + \Delta t'}{\sqrt{1-\beta^2}} \qquad (91b)$$

With $\Delta x' = 0$ in (91b) we get the equation for time dilation. With a rod of length $L = \Delta x$ in frame S, the length measured in S' requires a simultaneous measurement of the endpoints ($\Delta t' = 0$) and so from (91a) $L = L'/\sqrt{1-\beta^2}$ or $L' = L\sqrt{1-\beta^2}$, the equation for length contraction.

Example (Exercise 2.7.14). Substitute the transformation Equation (91) into the formula for the interval and verify that

$$(\Delta t)^2 - (\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 = (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2.$$

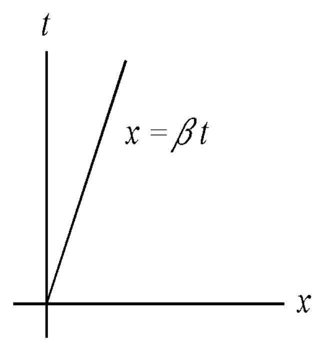
Solution. With $\Delta y = \Delta z = 0$ we have

$$\begin{aligned}
(\Delta t)^2 & -(\Delta x)^2 - (\Delta y)^2 - (\Delta z)^2 \\
= & \quad (\Delta t)^2 - (\Delta x)^2 = \left(\frac{\beta \Delta x' + \Delta t'}{\sqrt{1-\beta^2}}\right)^2 - \left(\frac{\Delta x' + \beta \Delta t'}{\sqrt{1-\beta^2}}\right)^2 \\
= & \quad \frac{\beta^2(\Delta x')^2 + 2\beta \Delta x' \Delta t' + (\Delta t')^2 - (\Delta x')^2 - 2\beta \Delta x' \Delta t' - \beta^2 (\Delta t')^2}{1-\beta^2} \\
= & \quad \frac{(\Delta x')^2(\beta^2 - 1) + (\Delta t')^2(1-\beta^2)}{1-\beta^2} \\
= & \quad (\Delta t')^2 - (\Delta x')^2 = (\Delta t')^2 - (\Delta x')^2 - (\Delta y')^2 - (\Delta z')^2
\end{aligned}$$

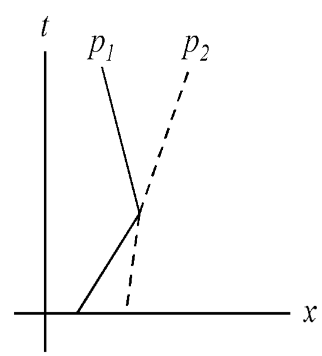
since $\Delta y' = \Delta z' = 0$.

2.8 Spacetime Diagrams

Note. We cannot (as creatures stuck in 3 physical dimensions) draw the full 4 dimensions of spacetime. However, for rectilinear or planar motion, we can depict a particle's movement. We do so with a spacetime diagram in which spatial axes (one or two) are drawn as horizontal axes and time is represented by a vertical axis. In the $xt-$plane, a particle with velocity β is a line of the form $x = \beta t$ (a line of slope $1/\beta$):



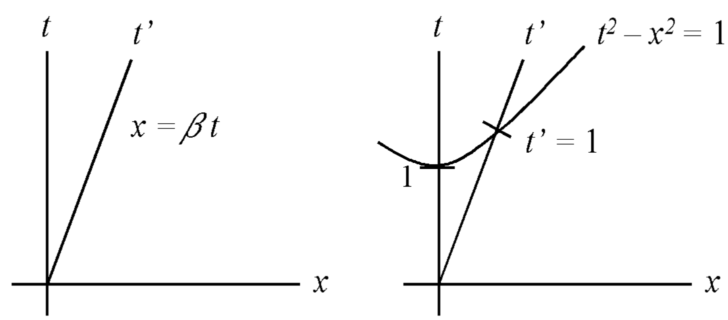
Two particles with the same spacetime coordinates must be in collision:



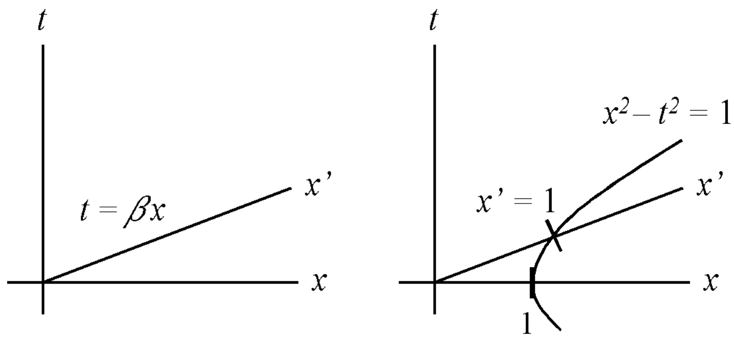
Note. The picture on the cover of the text is the graph of the orbit of the Earth as it goes around the Sun as plotted in a 3-D spacetime.

Definition. The curve in 4-dimensional spacetime which represents the relationships between the spatial and temporal locations of a particle is the particle's *world-line*.

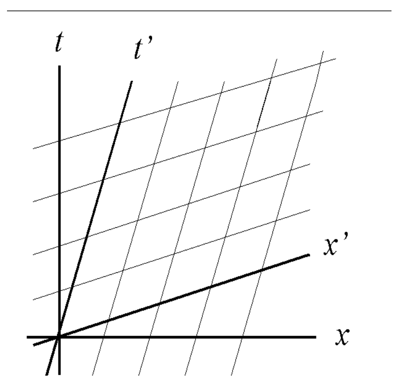
Note. Now let's represent two inertial frames of reference S and S' (considering only the $xt-$plane and the $x't'-$plane). Draw the x and t axes as perpendicular (as above). If the systems are such that $x = 0$ and $x' = 0$ coincide at $t = t' = 0$, then the point $x' = 0$ traces out the path $x = \beta t$ in S. We define this as the t' axis:



The hyperbola $t^2 - x^2 = 1$ in S is the same as the "hyperbola" $t'^2 - x'^2 = 1$ in S' (invariance of the interval). So the intersection of this hyperbola and the t' axis marks one time unit on t'. Now from equation (90b) (with $t' = 0$) we get $t = \beta x$ and define this as the x' axis. Again we calibrate this axis with a hyperbola ($x^2 - t^2 = 1$):



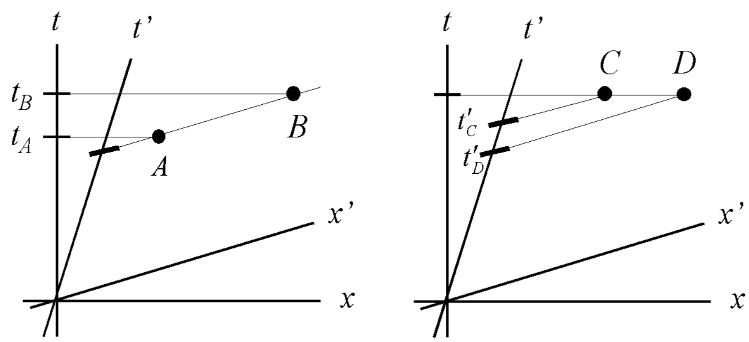
We therefore have:



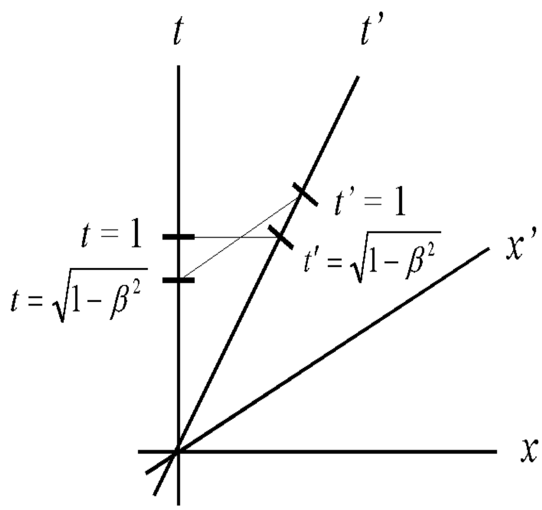
and so the S' coordinate system is oblique in the S spacetime diagram.

Note. In the above representation, notice that the larger β is, the more narrow the "first quadrant" of the S' system is and the longer the x' and t' units are (as viewed from S).

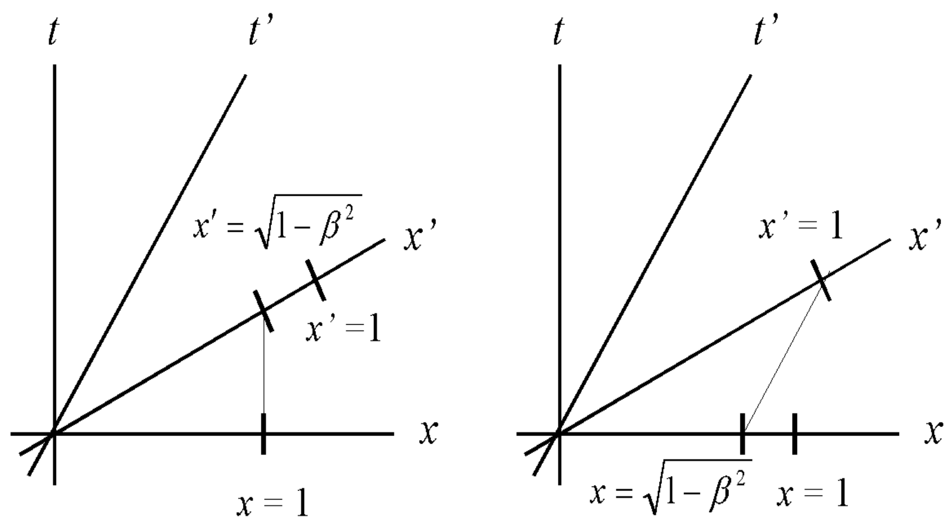
Note. Suppose events A and B are simultaneous in S' They need not be simultaneous in S. Events C and D simultaneous in S need not be simultaneous in S'.



Note. A unit of time in S is dilated in S' and a unit of time in S' is dilated in S.



Note. Suppose a unit length rod lies along the x axis. If its length is measured in S' (the ends have to be measured simultaneously in S') then the rod is shorter. Conversely for rods lying along the x' axis.



Example (Exercise 2.8.4). An athlete carrying a pole 16m long runs toward the front door of a barn so rapidly that an observer in the barn measures the pole's length as only 8m, which is exactly the length of the barn. Therefore at some instant the pole will be observed entirely contained within the barn. Suppose that the barn observer closes the front and back doors of the barn at the instant he observes the pole entirely contained by the barn. What will the athlete observe?

Solution. We have two events of interest:

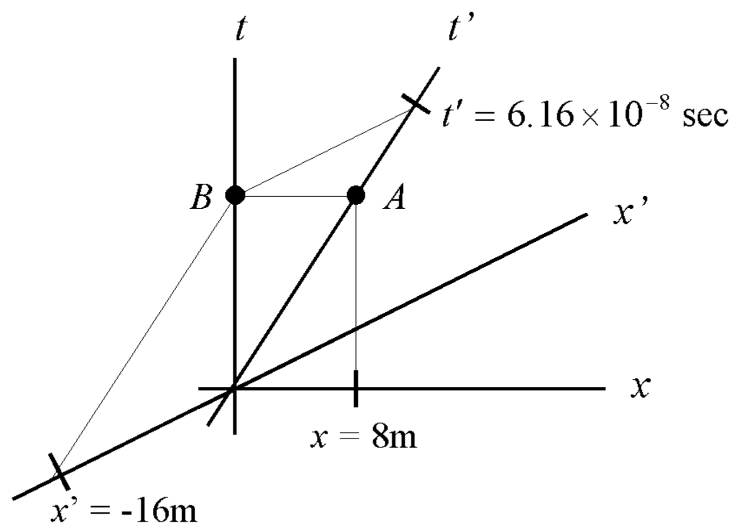
A = The front of the pole is at the back of the barn.

B = The back of the pole is at the front of the barn.

From Exercise 2.8.3, the observer in the barn (frame S) observes these events as simultaneous (each occuring at $t_{AB} = 3.08 \times 10^{-8}$sec after the front of the pole was at the front of the barn). However, the athlete observes event A after he has moved the pole only 4m into the barn. So for him, event A occurs when $t'_A = 1.54 \times 10^{-8}$sec. Event B does not occur until the pole has moved 16m (from $t' = 0$) and so event B occurs for the athlete when $t'_B = 6.16 \times 10^{-8}$sec. Therefore, the barn observer observes the pole totally within the barn (events A and B), slams the barn doors, and observes the pole start to break through the back of the barn all simultaneously. The athlete first observes event A along with the slamming of the back barn door and the pole starting to break through this door (when $t' = 1.54 \times 10^{-8}$sec) and THEN observes the front barn door slam at $t'_B = 6.16 \times 10^{-8}$sec. The spacetime diagram is:

A occurs at
$t' = 1.54 \times 10^{-8}$ sec

B occurs at
$t = 3.08 \times 10^{-8}$ sec

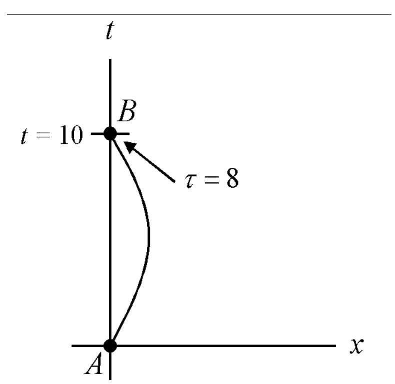


Since the order of events depends on the frame of reference, the apparent paradox is explained.

2.10 The Twin Paradox

Note. Suppose A and B are two events in spacetime separated by a timelike interval (whose y and z coordinates are the same). Joining these events with a straight line produces the world-line of an inertial observer present at both events. Such an observer could view both events as occuring at the same place (say at $x = 0$) and could put these two events along his $t-$axis.

Note. Oddly enough, in a spacetime diagram under Lorentz geometry, a straight line gives the longest distance (temporally) between two points. This can be seen by considering the fact that the interval $(\Delta\tau)^2 = (\Delta t)^2 - (\Delta x)^2$ is invariant. Therefore, if we follow a trajectory in spacetime that increases Δx, it MUST increase Δt. Figure II-19b illustrates this fact:



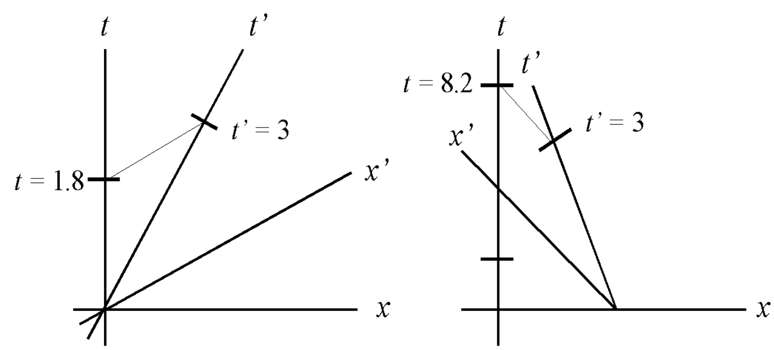
That is, the non-inertial traveler (the one undergoing accelerations and therefore the one not covered by special relativity) from A to B ages less than the inertial traveler between these two events.

Example (Jack and Jill). We quote from page 152 of the text: "Let us imagine that Jack is the occupant of a laboratory floating freely in intergalactic space. He can be considered at the origin of an inertial frame of reference. His twin sister, Jill, fires the engines in her rocket, initially alongside Jack's space laboratory. Jill's rocket is accelerated to a speed of 0.8 relative to Jack and then travels at that speed for three years of Jill's time. At the end of that time, Jill fires powerful reversing engines that turn her rocket around and head it back toward Jack's laboratory at the same speed, 0.8. After another three-year period, Jill returns to Jack and slows to a halt beside her brother. Jill is then six years older. We can simplify the analysis by assuming that the three periods of acceleration are so brief as to be negligible. The error introduced is not important, since by making Jill's journey sufficiently long and far, without changing the acceleration intervals, we could make the fraction of time spent in acceleration as small as we wish. Assume Jill travels along Jack's $x-$axis. In Figure II-20 (see below), Jill's world-line is represented on Jack's spacetime diagram. It consists of two straight line segments inclined to the $t-$axis with slopes $+0.8$ and -0.8, respectively. For convenience, we are using units of years for time and light-years for distance."

Note. Because of the change in direction (necessary to bring Jack and Jill back together), no single inertial frame exists in which Jill is at rest. But her trip can be described in two different inertial frames. Take the first to have t' axis $x = \beta t = 0.8t$ (in Jack's frame). Then at $t = 5$ and $t' = 3$, Jill turns and travels along a new t' axis of $x = -0.8t + 10$ (in Jack's frame). We see that upon the return, Jack has aged 10 years, but Jill has only aged 6 years. This is an example of the *twin paradox*.

Note. One might expect that the Principle of Relativity would imply that Jack should also have aged less than Jill (an obvious contradiction). However, due to the asymmetry of the situation (the fact that Jack is inertial and Jill is not) the Principle of Relativity does not apply.

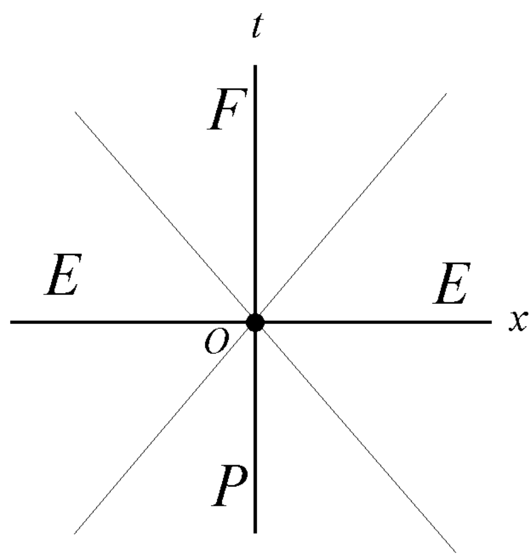
Note. Consider the lines of simultaneity for Jill at the "turning point":



So our assumption that the *effect of Jill's acceleration* is inconsequential is suspect! Jill's "turning" masks a long period of time in Jacks's frame ($t = 1.8$ to $t = 8.2$).

2.11 Temporal Order and Causality

Note. Suppose a flash of light is emitted at the origin of a spacetime diagram. The wavefront is determined by the lines $x = t$ and $x = -t$ where $t > 0$ (we use geometric units). We label the region in the upper half plane that is between these two lines as region F. Extending the lines into the lower half plane we similarly define region P. The remaining two regions we label E.



Note. Events in F are separated from O by a timelike interval. So O could influence events in F and we say O is *causally connected* to the events in F. In fact, if A is an event in the interior of F, then there is an inertial frame S' in which O and A occur at the same place. The separation between O and A is then only one of time (and as we claimed, O and A are separated by a timelike interval). The point A will lie in the "future" relative to O, regardless of the inertial frame. Therefore, region F is the *absolute future* relative to O.

Note. Similarly, events in P can physically influence O and events in P are causally connected to O. The region P is the *absolute past* relative to O.

Note. Events in region E are separated from O by a spacelike interval. For each event C in region E, there is an inertial frame S' in which C and O are separated only in space (and are simultaneous in time). This means that the terms "before" and "after" have no set meaning between O and an event in E. The region E is called *elsewhere.*

Note. We can extend these ideas and represent two physical dimensions and one time dimension. We then find the absolute future relative to an event to be a cone (called the *future light cone*). The *past light cone* is similarly defined. We can imagine a 4-dimensional version where the absolute future relative to an event is a sphere expanding in time.