

# Introduction to Math Philosophy and Meaning

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## INTRODUCTION

My definition of (pure) mathematics is: The use of logical principles to deduce implications (traditionally called *lemmas*, *theorems*, and *corollaries*) from a collection of consistent assumptions (called *postulates* or *axioms*). The axioms and theorems make claims about objects which are defined within the mathematical system. These objects, however, sometimes are informally defined — after all, we can only define things in terms of other things. To avoid circular definitions, we must take some definitions as intuitive and fundamental (this is often the case with such objects as “sets” and “points”), but this is to be kept to a minimum and the properties of these objects are elaborated in the axioms and theorems.

My definition would fall under the “metamathematical” view (i.e., view *about* math) known as *formalism*. Formalism is “the assertion of the possibility and desirability of banishing intuitions by showing formal systems to be entirely adequate to the business of mathematics” [Goldstein, page 134]. This view of math was championed by David Hilbert.

## DAVID HILBERT



David Hilbert (1862–1943)

Hilbert was interested in stripping math (geometry in particular) of all intuition — he wanted to make it simply the manipulation of symbols according to certain logical rules. He stated that “mathematics is a game played according to certain simple rules with meaningless marks on paper” [Goldstein, page 136]. Another famous quote attributed to Hilbert is: “One must be able to say at all times — instead of points, straight lines, and planes — tables, chairs, and beer mugs.” Hilbert’s point is that the axiomatic systems of mathematics are not *about* anything in the “real world.” In a sense, they are worlds unto themselves and their only existence is in our minds (personally, I largely agree with this philosophy of mathematics).

To put things in a historical context, Hilbert’s work on this subject takes place around 1900. This is some 70–80 years after the “discovery” (or is it an “invention”?) of non-Euclidean geometry. It was thought for over 2000 years that the

only logical and “true” geometry was that of Euclid. So the discovery of alternative geometries in the early 1800’s were a great surprise (and the source of some controversy). But geometry wasn’t the only area of mathematics experiencing a revolution. William Rowan Hamilton (1805–65) created the quaternions (the first example of a non-commutative algebra) in the 1840s. This result, along with the recent acceptance of complex numbers, lead to a liberation of algebra from thinking of numbers in terms of absolute rules. It was realized that “the *rules* of algebra could be formulated independently of the *objects* to which they were applied; this discovery set algebra free to invent and study new structure, and a very rapid development set in around the middle of the nineteenth century” [Sondheimer and Rogerson, page 75]. This rapid development included the study of abstract groups, vector spaces, and matrices. The resulting philosophical implication is that the objects of math are maybe not absolutely determined by some intuitive constraints — maybe they are simply the results of certain almost arbitrary assumptions.

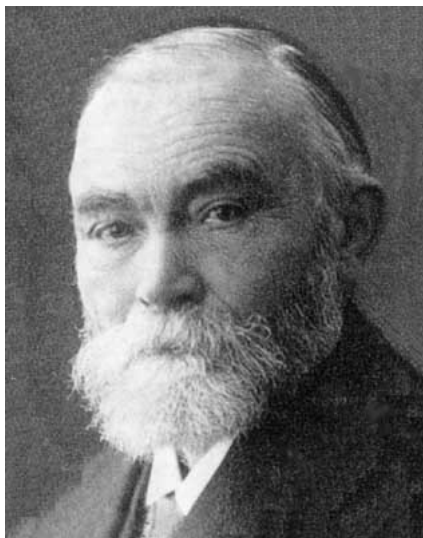
Some of the geometric proofs of Euclid’s *Elements* are not entirely adequate by “modern” mathematical standards (there are unstated assumptions concerning lines, for example). So to sort of “clean up” Euclid, Hilbert published *Foundations of Geometry* in 1899 (an online PDF copy can be found on the [Project Gutenberg webpage](#) (accessed 6/25/2022)). This is “said to have been the most influential work in geometry since Euclid” [Goldstein, page 137]. This is quite a claim since the work on the foundations of non-Euclidean geometry had appeared a few decades earlier!

In 1900, Hilbert gave a talk at the Second International Congress of Mathematicians in which he tried to set the tone of math for the 20th century. Hilbert’s

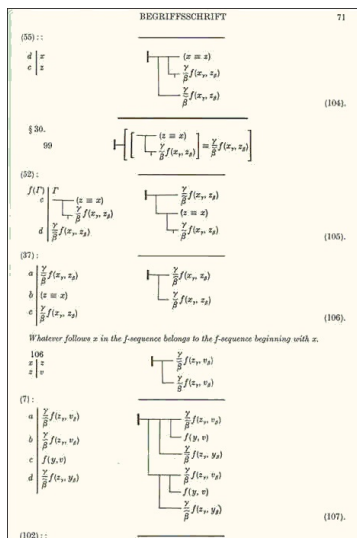
attitude at the time seems to be that the proofs of the consistency of geometry and arithmetic were imminent. It appeared that, after a few more details were ironed out, math would become a clean rigorous, totally-unintuitive endeavor! But... some of the details were more complicated than they appeared on the surface. (Coincidentally, physics of the time was in much the same situation...just before the revolutions of relativity and quantum mechanics.)

Enter Gottlob Frege and Bertrand Russell.

## FREGE, RUSSELL, AND *PRINCIPIA*



Gottlob Frege (1848–1925)



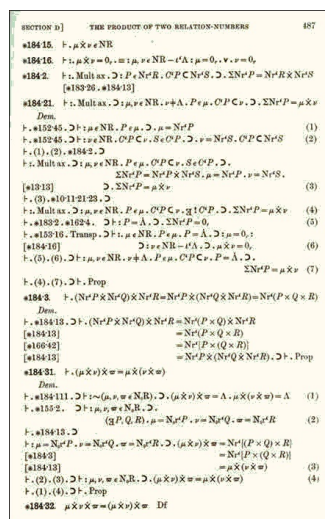
A page from *Fundamental Laws of Arithmetic*

Two monumental attempts were made around 1900 to base all of mathematics (starting with arithmetic) on logic and set theory. In 1893 Gottlob Frege published *Fundamental Laws of Arithmetic*. A more famous work is due to Bertrand Russell and Alfred North Whitehead called *Principia Mathematica* (1910) (an online copy

is available on the [University of Michigan Historical Math Collection webpage](#) (accessed 6/25/2022). The level of detail was so great that it took over 200 pages to prove that  $1 + 1 = 2$ . The plan was to base arithmetic on set theory — after all, what could be more reliable than set theory?



Bertrand Russell (1872–1970)



A page from *Principia*

It turns out that the set theory of 1900 wasn't as flawless as was assumed! Russell himself was instrumental in finding a crack in the foundations. One way to explain Russell's Paradox is as follows:

Imagine a town with a barber. The barber cuts the hair of all of those who do not cut their own hair. Who cuts the barber's hair? If he does not cut his own hair, then he *must* cut his own hair (since that is his job). If he does cut his own hair, then he cannot cut his own hair since his job is to cut the hair of those who do not cut their own hair.

Another more set theoretic way is to consider the set  $E$  of all sets that are not members of themselves. The question then is: Is set  $E$  a member of itself? If  $E$  is a member of itself, then it cannot be a member of itself since it only consists of

such sets. If  $E$  is not a member of itself, then it must be a member of itself by its own definition. Therefore such a set cannot exist. The moral of Russell's Paradox is that you cannot just define a set any old way you want! This led to a revision of axiomatic set theory and a specific protocol by which new sets could be defined in terms of existing sets. With these slight changes, the foundation of set theory should be solid, and the work of building mathematical knowledge could go on. Presumably, the formalist's dream of having an axiomatic system within which all "mathematical truths" could be derived was on the horizon.

Enter Kurt Gödel.

## KURT GODEL



Kurt Gödel (1906–1978)

In 1931, a 25 year old mathematician at the University of Vienna published a 26 page paper titled "On Formally Undecidable Propositions of *Principia Mathematica* and Related Systems." Initially there was not widespread appreciation of

the importance of this research, but its significance would become very well-known. Gödel addressed two properties of axiomatic systems: consistence and completeness. *Consistence* means that no contradiction can be derived within the system. Completeness is a bit harder to explain.

The rules of manipulation in a formal axiomatic system are of these sorts: (1) the rules that specify what the symbols are (the “alphabet” of the system), (2) the rules that specify how the symbols can be put together to make “well-formed formulas” (or ‘WFF’s) which include the claims of the system (the lemmas, corollaries, and theorems), and (3) the rules of inference that specify which WFFs can be derived from other WFFs [Goldstein, page 86]. An axiomatic system is *complete* if a truth value can be put on every WFF. What this means is that every meaningful statement can either be proven to be true or false. (In geometry, examples of WFFs are: “Two lines, parallel to a third, are parallel to each other” [true in Euclidean geometry], and “The sum of the measures of the angles of a triangle is less than  $180^\circ$ ” [false in Euclidean geometry]. An example of a statement which involves the objects of geometry, but is not a WFF [since it is meaningless] is: “All points are parallel.”)

Gödel’s two main results are:

**Gödel’s First Incompleteness Theorem.** There are provably unprovable but nonetheless true propositions in any formal system that contains elementary arithmetic, assuming that system to be consistent.

**Gödel’s Second Incompleteness Theorem.** The consistency of a formal system that contains arithmetic can’t be formally proved with that system [Goldstein, page 183].

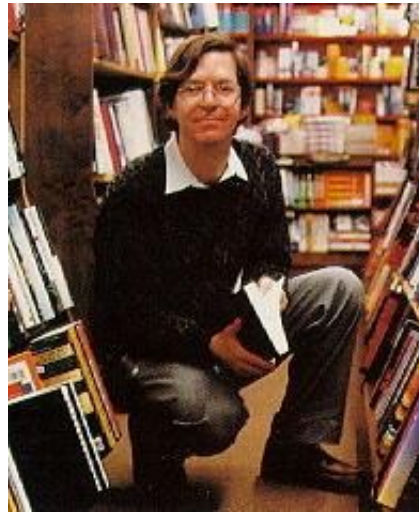
What Gödel *has* shown is that there are meaningful statements in axiomatic systems (which include arithmetic) which can neither be proved to be true nor proved to be false. Such statements are said to be *undecidable*. A specific example of this is the *Continuum Hypothesis* which addressed the existence of a set of real numbers  $S$  such that  $|\mathbb{N}| < |S| < |\mathbb{R}|$ . This idea of undecidability was disturbing to a number of pure mathematicians of the time. After all, these results, if not contradicting the work of Frege and Russell, is in direct contrast to the *spirit* of these works. However, other mathematicians took the results in stride with the attitude that mathematics will continue to go forward and undecidable propositions are now just part of the terrain.

What Gödel *did not show* was that all formal axiomatic systems are incomplete. In fact, Gödel's Ph.D. dissertation was on the completeness of a system called "predicate calculus." Also, Gödel does not say that the consistency of a formal system of arithmetic cannot be proved by any means; only that it cannot be proved within the system itself. Gödel also said nothing about sociological or cultural ideas. However, there is a movement (which I will refer to as "radical postmodernism") to employ the ideas from mathematics and physics (in particular, relativity and quantum mechanics) in the study of societies, cultures, and politics. This reflects a profound misunderstanding of the meaning of the mathematical and physical statements. As mentioned above, Gödel does not propose incompleteness/undecidability in *all* settings (not even in all mathematical settings)! Einstein's theory of relativity does not imply that "everything is relative" (in fact, the special theory of relativity deals explicitly with a quantity that is an absolute between different frames of reference). Quantum theory only deals with



the atomic level and certainly not the societal level (and even at the atomic level, it does not necessarily imply random behavior — certain interpretations of quantum theory are still deterministic). In response to the radical postmodernist, Sokal and Bricmont [page  $x$ ] state:

“We are sometimes accused of being arrogant scientists, but our view of the hard sciences’ role is in fact rather modest. Wouldn’t it be nice (for us mathematicians and physicists, that is) if Gödel’s theorem or relativity theory *did* have immediate and deep implications for the study of society? Or if the axiom of choice could be used to study poetry? Or if topology had something to do with the human psyche? But alas, it is not the case.”



Alan Sokal

Alan Sokal is famous (or infamous) for publishing an article “Transgressing the Boundaries: Towards a Transformative Hermeneutics of Quantum Gravity” in the research journal *Social Text* in 1996. Sokal had witnessed the proliferation of studies making bizarre applications of math and physics, as mentioned above. In his

paper, meant as a parody of this type of work, Sokal claims that there is no physically real world [Sokal and Bricmont, page 2]: “physical ‘reality’, no less than social ‘reality’, is at bottom a social and linguistic construct.” His arguments, intentionally nonsensical, passed the refereeing process of the cultural-studies journal *Social Text* and was even published in a special issue devoted to rebutting the criticisms of postmodernism and social constructivism. Many researchers in the humanities and social sciences wrote to Sokal to thank him for what he had done. Others were not so happy!

## PEANO’S AXIOMS OF ARITHMETIC



Giuseppe Peano (1858–1932)

For the sake of illustration, let’s look at an axiomatic system for elementary arithmetic. In 1899, the Italian mathematician Giuseppe Peano reduced arithmetic to five axioms [Nagel and Newman, page 114]:

**Axiom 1.** 0 is a number.

**Axiom 2.** The immediate successor of a number is a number.

**Axiom 3.** 0 is not the immediate successor of a number.

**Axiom 4.** No two numbers have the same immediate successor.

**Axiom 5.** Any property belonging to 0, and also to the immediate successor of every number that has the property, belongs to all numbers.

What Peano is doing here is actually defining the nonnegative integers  $\{0, 1, 2, 3, \dots\}$ .

Axiom 5 is called the Principle of Mathematical Induction.

Recall that a positive integer is *prime* if it's only factors are 1 and itself. The first several prime numbers are 2, 3, 5, 7, 11, 13, 17, 19, 23, 29, 31, 37. A couple of statements from elementary arithmetic concerning prime numbers are:

**Twin Prime Conjecture.** There are an infinite number of twin primes. (Two primes are *twin primes* if they differ by 2 — for example, 3 and 5, 5 and 7, 11 and 13,  $6797227 \times 2^{15328} - 1$  and  $6797227 \times 2^{15328} + 1$  [T. Forbes, “A Large Pair of Twin Primes,” *Math. Comp.* **66** (1997), 451–455]). The current record holder seems to be:  $3756801695685 \times 2^{666669} - 1$  and  $3756801695685 \times 2^{666669} + 1$  (which have 200,700 digits— see [University Tennessee at Martin webpage](#) (accessed 6/25/2022)).

**The Goldbach Conjecture.** Every even integer greater than or equal to 4 can be written as the sum of two primes. This has been verified up to around  $4 \times 10^{18}$ , according to [Tomás Oliveira e Silva webpage](#) (accessed 6/25/2022).

So far, there is no proof for either of these conjectures. Nor are there proofs of their negations (a counterexample would do for this in the case of the negation of the Goldbach Conjecture). Are these results true, false, or undecidable?

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5. [MacTutor History of Mathematics Archive](#) for pictures of Frege, Gödel, Hilbert, Peano, and Russell.
6. [Stephen Wolfram webpage](#) for pictures of pages from *Arithmetica* and *Principia*.
7. [Alan Sokal's webpage](#) for the picture of Alan Sokal.

Websites were accessed July 25, 2013.

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