Computational Complexity

Chapter 3. Forms

3.1. σ-Sesquilinear Forms—Proofs of Theorems

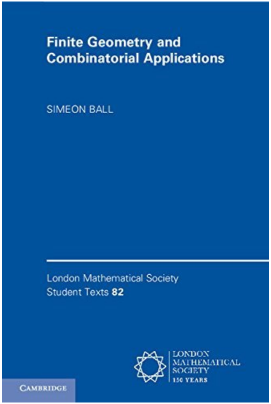

Table of contents

Lemma 3.1

Lemma 3.1. Let U be a subspace of $\mathsf{V}_k(\mathbb{F})$. If b is a non-degenerate σ-sesquilinear form on $\mathsf{V}_k(\mathbb{F})$ then $\dim\ U + \dim\ U^{\perp} = k$.

Proof. Let $\{e_1, e_2, \ldots, e_r\}$ be a basis for U. Define linear maps α_i for $i = 1, 2, \ldots, r$ as $\alpha_i(v) = b(e_i, v)$ (if b is an inner product, then $\alpha_i(v)$ is the projection of v onto basis vector e_i).

Lemma 3.1

Lemma 3.1. Let U be a subspace of $\mathsf{V}_k(\mathbb{F})$. If b is a non-degenerate σ-sesquilinear form on $\mathsf{V}_k(\mathbb{F})$ then $\dim\, U + \dim\, U^{\perp} = k$.

Proof. Let $\{e_1, e_2, \ldots, e_r\}$ be a basis for U. Define linear maps α_i for $i = 1, 2, \ldots, r$ as $\alpha_i(v) = b(e_i, v)$ (if b is an inner product, then $\alpha_i(v)$ is the projection of v onto basis vector e_i).

If $\sum_{i=1}^{n} \lambda_i \alpha_i = 0$ then, by definition, $\sum_{i=1}^{r} \lambda_i \alpha_i(v) = 0$ for all $v \in \mathsf{V}_k(\mathbb{F})$. Therefore, because b is linear in the first position,

$$0 = \sum_{i=1}^{r} \lambda_i \alpha_i(v) = \sim_{i=1}^{r} \lambda_i b(e_i, v) = b\left(\sum_{i=1}^{n} \lambda_i e_i, v\right)$$

and so $\sum_{i=1}^{r} \lambda_i e_i = 0$ since b is nondegenerate by hypothesis. Since $\{e_1, e_2, \ldots, e_r\}$ be a basis for U, then the e_i are linearly independent and so we must have $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$. This implies that $\alpha_1, \alpha_2, \ldots, \alpha_r$ are linearly independent.

Lemma 3.1

Lemma 3.1. Let U be a subspace of $\mathsf{V}_k(\mathbb{F})$. If b is a non-degenerate σ-sesquilinear form on $\mathsf{V}_k(\mathbb{F})$ then $\dim\ U + \dim\ U^\perp = k$.

Proof. Let $\{e_1, e_2, \ldots, e_r\}$ be a basis for U. Define linear maps α_i for $i = 1, 2, \ldots, r$ as $\alpha_i(v) = b(e_i, v)$ (if b is an inner product, then $\alpha_i(v)$ is the projection of v onto basis vector e_i).

If $\sum_{i=1}^n \lambda_i \alpha_i = 0$ then, by definition, $\sum_{i=1}^r \lambda_i \alpha_i(v) = 0$ for all $v \in \mathsf{V}_k(\mathbb{F})$. Therefore, because b is linear in the first position,

$$0 = \sum_{i=1}^r \lambda_i \alpha_i(v) = \sim_{i=1}^r \lambda_i b(e_i, v) = b\left(\sum_{i=1}^n \lambda_i e_i, v\right)$$

and so $\sum_{i=1}^r \lambda_i e_i = 0$ since b is nondegenerate by hypothesis. Since $\{e_1, e_2, \ldots, e_r\}$ be a basis for U, then the e_i are linearly independent and so we must have $\lambda_1 = \lambda_2 = \cdots = \lambda_r = 0$. This implies that $\alpha_1, \alpha_2, \ldots, \alpha_r$ are linearly independent.

Theorem 4.1

Theorem 4.1. Every Boolean expression is equivalent to one in conjunctive normal form, and to one in disjunctive normal form.

Proof.

Theorem 4.1

Theorem 4.1. Every Boolean expression is equivalent to one in conjunctive normal form, and to one in disjunctive normal form.

Proof.