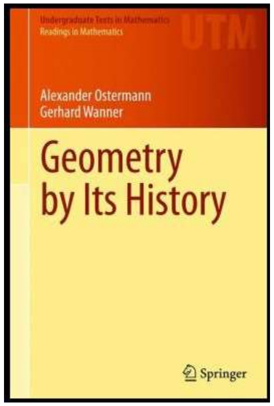
History of Geometry

Chapter 2. The Elements of Euclid

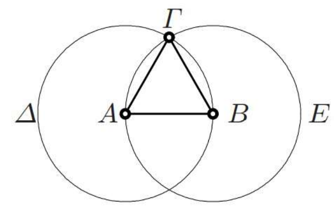
2.1. Book I—Proofs of Theorems



Euclid, Book I Proposition 1

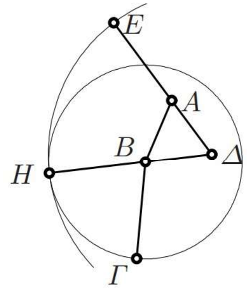
Euclid, Book I Proposition 1. On a given finite straight line AB to construct an equilateral triangle.

Proof. By Postulate 3, we can construct a circle Δ centered at point A and passing through point B. Also by Postulate 3, construct a circle E centered at point B and passing through point A. Let Γ be a point of intersection of circles Δ and E. Construct a line segment joining points Γ and A, and construct a line segment joining Γ and B (using Postulate 1). Then the distance $A\Gamma$ is equal to $B\Gamma$ and equal to AB (by the definition of circle, Book I Definition 15). □



Euclid, Book I Proposition 2

Euclid, Book I Proposition 2. To place at a given point A a straight line AE equal to a given straight line $B\Gamma$.



Proof. Use Euclid I.1 to construct an equilateral triangle $AB\Delta$ on segment AB. By Postulate 3, construct the circle with center B and passing through point Γ. Extend line segment ΔB from point B until it intersects this circle at, say, point H (Postulate 2 allows this). Also by Postulate 3, construct a second circle with center Δ passing through H. Extend line segment ΔA from point A until it intersects this second circle at, say, point E (Postulate 2). The distance $B\Gamma$ equals the distance BH (because of properties of the first circle) and distance ΔH equals distance ΔE (because of properties of the second circle). So $\Delta E = \Delta A + AE = \Delta H = \Delta B + BH$ or $AE = BH$, since $\Delta B = \Delta A$. □

Euclid, Book I Proposition 5

Euclid, Book I Proposition 5. If in a triangle, $a = b$, then $\alpha = \beta$.

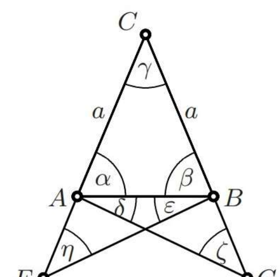


Figure 2.2(a)

Proof. Here, we present Euclid's proof and consider Figure 2.2(a). By Postulate 2 we can extend line segments CA and CB "continuously" to points F and G such that, by Euclid I.2, $AF = BG$ (without loss of generality, $CA < CB$ and we can extend CA and produce a segment CF longer than CB and then we can extend CB to produce a segment CG equal in length to CB by Euclid I.2; from this we have $AF = BG$). Next, by Postulate 2 we can introduce segments FB and AG. Now by Euclid I.4 ("side-angle-side"), triangles FCB and GCA are equal. So, corresponding angles and sides of the triangles are equal, so that $\alpha + \delta = \beta + \varepsilon$, $\eta = \zeta$, and $FB = GA$. Again by Euclid I.4, triangle AFB and BGA are equal and so $\delta = \varepsilon$. Since $\alpha + \delta = \beta + \varepsilon$ and $\delta = \varepsilon$, then $\alpha = \beta$, as claimed. □

Euclid, Book I Proposition 7

Euclid, Book I Proposition 7. Consider the two triangles of Figure 2.3(a), with the same base AB and with the third vertex on the same side of the base. If $a = a'$ and $b = b'$, then points C and D are the same, $C = D$.

Proof. We present Euclid's proof. ASSUME $C \neq D$. Since triangle DAC is isosceles by hypothesis, then by Euclid I.5 we have $\alpha + \beta = \gamma$. Similarly, triangle DBC is isosceles and, again by Euclid I.5, $\beta = \gamma + \delta$. Since $\alpha > 0$ then $\gamma > \beta$, and since $\delta > 0$ then $\gamma < \beta$, a CONTRADICTION. So the assumption that $c \neq D$ is false, and hence $C = D$ as claimed. □

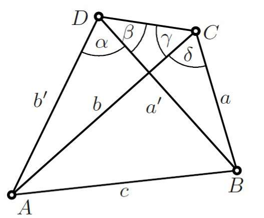


Figure 2.3(a)

Euclid, Book I Proposition 8

Euclid, Book I Proposition 8. If two triangles ABC and DEF have sides of equal lengths, then they also have equal angles.

Proof. Ostermann and Wanner credit the proof they give to Philo of Byzantium (circa 280 BCE–circa 220 BCE), declaring it "more elegant" than Euclid's proof (though they do not give a specific reference). We move triangle ABC onto triangle DEF in such a way that the line segment BC is placed on segment EF (this is possible since these are the same length) and the point A is moved onto point G on the opposite side of EF to D; see Figures 2.3(b) and (c).

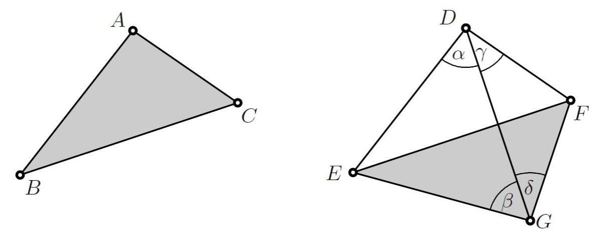


Figure 2.3(b) and (c)

Euclid, Book I Proposition 8 (continued)

Euclid, Book I Proposition 8. If two triangles ABC and DEF have sides of equal lengths, then they also have equal angles.

Proof (continued). So $DE = DG$ and triangle DEG is isosceles. Hence, by Euclid I.5, $\alpha = \beta$. Similarly, triangle DFG is also isosceles and $\gamma = \delta$. So $\beta + \delta = \alpha + \gamma$; that is, the angle at A (equal to $\beta + \delta$) equals the angle at D (equal to $\alpha + \gamma$). Similar arguments (with different movements) show the other corresponding angles are also equal. □

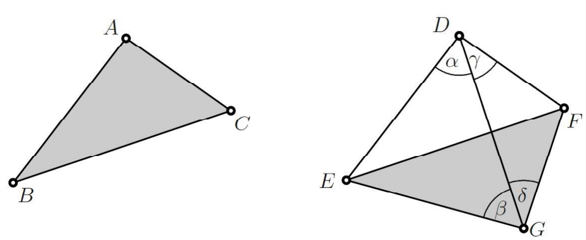


Figure 2.3(b) and (c)

Euclid, Book I Proposition 13

Euclid I.13. Let the line AB cut the line CD. With α and β as the two resulting angles on the same side of line CD, we have $\alpha + \beta = 2\llcorner$.

Proof. Let B be the point of intersection of the lines. If the lines are perpendicular, then $\alpha = \beta = \llcorner$ and we are done. So without loss of generality, we may assume that one of α and β is greater than $\llcorner$, say $\beta > \llcorner$. Euclid I.11 gives the existence of a perpendicular to CD through point B. With η as the angle between this perpendicular and line AB, we have that $\beta = \llcorner + \eta$ and $\alpha + \eta = \llcorner$ (see Figure 2.5). Combining these two equations gives $\alpha + \eta + \beta = 2\llcorner + \eta$, or $\alpha + \beta = 2\llcorner$, as claimed. □

Figure 2.5

Euclid, Book I Proposition 14

Euclid, Book I Proposition 14. Let line segment *DB* and line segment *BA* determine an angle β. If segment *BC* makes an angle α with segment *BA* where point *C* is exterior to the first angle. If $\alpha + \beta = 2\llcorner$ then *C* lies on the line *DB*.

Proof. Let *E* line on the line *DB* and let γ be the angle between segments *BA* and *BE*. Then by Euclid I.13, $\gamma + \beta = 2\llcorner$. By hypothesis, $\alpha + \beta = 2\llcorner$, so $\gamma + \beta = \alpha + \beta$ (by Postulate 4) and hence $\gamma = \alpha$. So points *E* and *C* lie on the same line (i.e., lie on the same side of the common angle). Since *E* lies on line *DB* then we have that *C* lies on line *DB*, as claimed. □

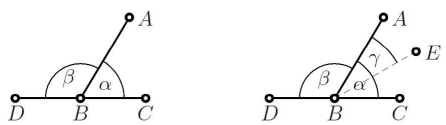


Figure 2.6

Euclid, Book I Proposition 15

Euclid, Book I Proposition 15. If two lines cut one another, they make the opposite angles equal to one another.

Proof. Consider the opposite angles α and β, and introduce angle γ as given in Figure 2.7. By Euclid I.13 we have $\alpha + \gamma = 2\llcorner$ and $\gamma + \beta = 2\llcorner$. So $\alpha + \gamma = \gamma + \beta$ (by Postulate 4). Therefore $\alpha = \beta$, as claimed. □

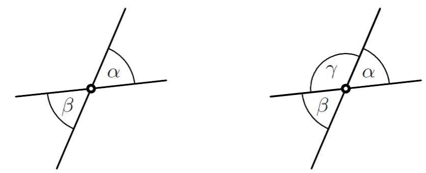


Figure 2.7

Euclid, Book I Proposition 16

Euclid, Book I Proposition 16. If one side of a triangle is extended at *C*, the exterior angle is greater than both angles in the triangle opposite to *C*.

Proof. Let *E* be the midpoint of *AC*, which can be found by Euclid I.10. By Postulate 1, line segment *BE* exists. By Postulate 2, line segment *BE* can be extended to a point *F* such that $BE = EF$. The "grey" angles at point *E* of Figure 2.8 are opposite angles and so by Euclid I.15 are equal. So by Euclid I.4 (side-angle-side) the two grey triangles are equal. Therefore the grey angle at point *C* is α and is "obviously" smaller than δ (we could use Common Notion 5 here); that is, $\delta > \alpha$, as claimed. Similarly, by bisecting side *BC* and introducing a triangle with a vertex at point *C*, we can use the other angle of size δ to show $\delta > \beta$, as claimed. □

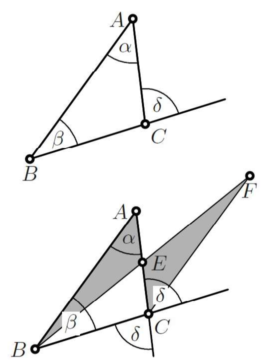


Figure 2.8

Euclid, Book I Proposition 27

Euclid, Book I Proposition 27. If some line cuts two line *a* and *b* such that alternate interior angles α and β are equal, then lines *a* and *b* are parallel, denoted $a \parallel b$.

Proof. Let points *E* and *F* be the points of intersection of the cutting line with lines *a* and *b* (see Figure 2.10). ASSUME that lines *a* and *b* are not parallel; then they meet at some point *G*. Without loss of generality, suppose point *G* is on side of the cutting line in which angle β lies, as in Figure 2.10. Then *EGF* is a triangle with α as an exterior angle. So by Euclid I.16, $\alpha > \beta$. But this CONTRADICTS the assumption that $\alpha = \beta$. So the assumption that lines *a* and *b* are not parallel is false, and hence lines *a* and *b* are parallel, as claimed. □

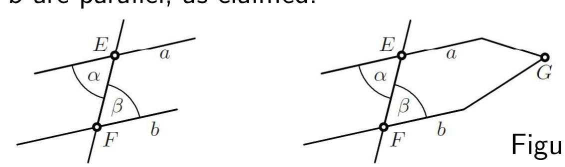


Figure 2.10

Euclid, Book I Proposition 29

Euclid, Book I Proposition 29. Parallel lines cut by some line, have alternate interior angles are equal.

Proof. Let *a* and *b* be parallel lines and let α and β be alternate interior angles that result from the cutting line (see Figure 2.11). ASSUME $\alpha > \beta$. Introduce angle γ as in Figure 2.11. By Euclid I.13 we have $\alpha + \gamma = 2 ⊾$, and hence $\beta + \gamma < 2 ⊾$. By the Parallel Postulate (Postulate 5), lines *a* and *b* meet (and do so on the side of the cutting line containing angles β and γ), CONTRADICTING the hypothesis that $a \parallel b$. So the assumption that $\alpha > \beta$ is false and hence we have $\alpha \leq \beta$. Similarly, if we assume $\alpha < \beta$ we can show that $\alpha \geq \beta$. Hence $\alpha = \beta$, as claimed. □

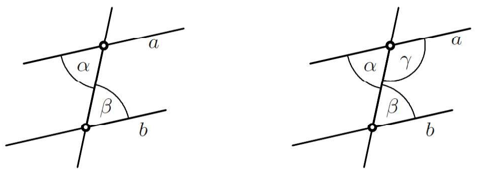


Figure 2.11

Euclid, Book I Proposition 30

Euclid, Book I Proposition 30. For any three (distinct) lines *a*, *b*, *c*, if $a \parallel b$ and $b \parallel a$ then $a \parallel c$.

Proof. Let the angles between the parallel lines *a*, *b*, *c* and the cutting line be α, β, γ, respectively (see Figure 2.12). If we introduce the angles opposite α, β, γ as α', β', γ' (not pictured in Figure 2.12) then by Euclid I.15 we have $\alpha = \alpha'$, $\beta = \beta'$, and $\gamma = \gamma'$. By Euclid I.29, $\alpha = \beta'$ since $a \parallel b$, and $\beta = \gamma'$ since $b \parallel c$. Hence, $\alpha = \beta' = \beta = \gamma'$. Finally, by Euclid I.27, we have $a \parallel c$, as claimed. □

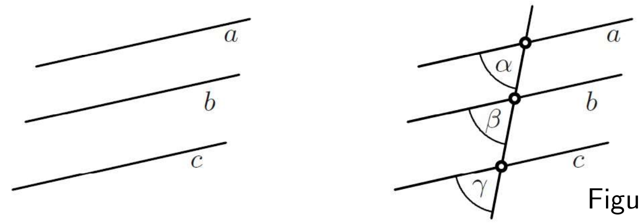


Figure 2.12

Euclid, Book I Proposition 31

Euclid, Book I Proposition 31. To draw a parallel to a given line *a* through a given point *A* not on the *a*.

Proof. By Euclid I.12, we can construct a perpendicular to line *a* through point *A*. By Euclid I.11, we can construct a perpendicular *b* to the perpendicular through point *A*. We then have that the alternate interior angles for lines *a* and *b* are both right angles and hence by Euclid I.27 line *b* is parallel to line *a* and passes through point *A*, as needed. □

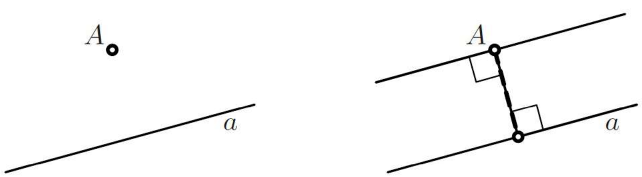


Figure 2.13