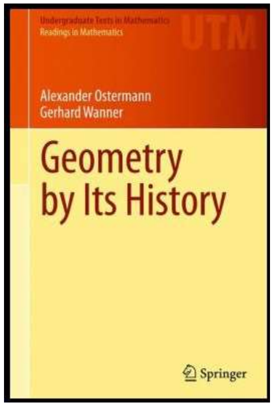
History of Geometry

Chapter 2. The Elements of Euclid

2.2. Book III. Properties of Circles and Angles—Proofs of Theorems



Euclid, Book III Proposition 22

Euclid, Book III Proposition 22. Let *ABCD* be a quadrilateral inscribed in a circle. Then the sum of two opposite angles equals two right angles: $\alpha + \delta = 2\llcorner$.

Proof. Consider the triangle *ABC* in Figure 2.15(b). By Euclid III.21, we have the angle β at point *D* as given (think of sliding point *B* around to point *D*, and we have angle γ at point *D* as given (thin of sliding point *C* around to point *D*). So we have $\delta = \beta + \gamma$. Euclid I.32 states that the sum of the angles of a triangle is equal to two right angles, so in triangle *ABC* we have $\alpha + \beta + \gamma = 2\llcorner$. Hence, $\alpha + \delta = 2\llcorner$, as claimed. □

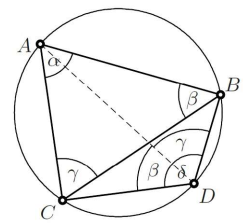


Figure 2.15(b)

Euclid, Book III Proposition 35

Euclid, Book III Proposition 35. If two chords *AB* and *CD* of a circle intersect in a point *E* inside the circle, then $AE \cdot EB = CE \cdot ED$.

Proof. Euclid's (lengthy) proof avoids the use of Thales' Theorem (Theorem 1.1). Instead, we claim that triangles *AEC* and *DEB* are similar, and then use Thale's Theorem. The opposite angles labeled ε in Figure 2.16(b) are equal by Euclid I.15. The angles at points *A* and *D* are equal (labeled α in Figure 2.16(b)) by Euclid III.21 (consider them as angles in triangles *CAB* and *CDB* with common base *CB*) and, similarly, the angles at points *C* and *B* are equal. So the *AEC* and *DEB* are similar, as claimed, and by Thale's Theorem $AE/CE = ED/EB$, or $AE \cdot EB = CE \cdot ED$, as claimed □

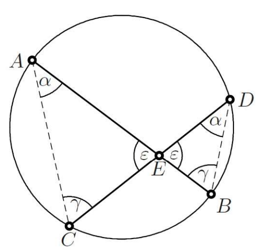


Figure 2.16(b)

Euclid, Book III Proposition 36

Euclid, Book III Proposition 36. Let *E* be a point outside a circle and consider a line through *E* that cuts the circle in two points *A* and *B*. Further let *T* be the point of tangency of a tangent through *E* (see Figure 2.17(a)). Then $AE \cdot BE = (TE)^2$.

Proof. The two angles labeled α in Figure 2.17(a) are equal by Euclid III.21 and Exercise 2.17. In triangle *ATE* the angle at *A* corresponds to the angle at *T* in triangle *TBE*, and both of these angles are α. In triangle *ATE* the angle at *E* corresponds to the angle at *E* in triangle *TBE* and this common angle is ε.

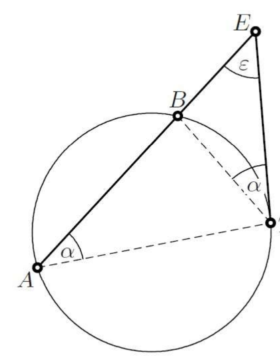


Figure 2.17(a)

Euclid, Book III Proposition 36

Euclid, Book III Proposition 36. Let E be a point outside a circle and consider a line through E that cuts the circle in two points A and B. Further let T be the point of tangency of a tangent through E (see Figure 2.17(a)). Then $AE \cdot BE = (TE)^2$.

Proof (continued). The angle sum of a triangle is 2 ⦜ by Euclid I.32 then, since in triangle ATE the angle at T corresponds to the angle at B in triangle TBE and both of these angles must be equal. That is, triangles ATE and TBE are similar. So by Thales' Theorem (Theorem 1.1), $AE/TE = TE/BE$ or $AE \cdot BE = (TE)^2$, as claimed. □

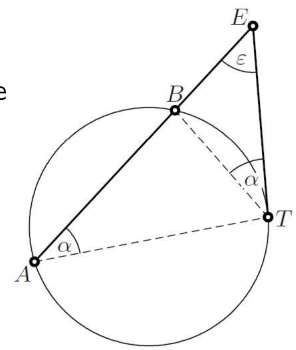


Figure 2.17(a)

Corollary (Clavius 1574)

Corollary (Clavius 1574). Let A, B, C, and D denote four points on a circle. If the line AB meets the line CD in a point E outside the circle (see Figure 2.17(b)), then $AE \cdot BE = CE \cdot DE$.

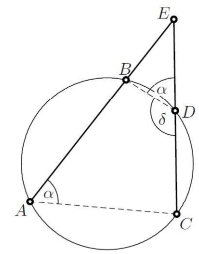


Figure 2.17(b)

Proof. By Euclid III.36, $AE \cdot BE = (TE)^2$ where T is the point of tangency of a tangent through E. Similarly, Euclid III.36 implies that $CE \cdot DE = (TE)^2$. Therefore, $AE \cdot BE = DE \cdot DE$, as claimed. □