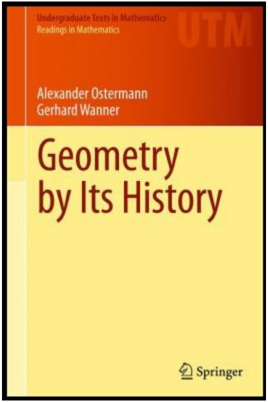
History of Geometry

Chapter 2. The Elements of Euclid

2.2. Book III. Properties of Circles and Angles—Proofs of Theorems



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Proof. Consider the triangle *ABC* in Figure 2.15(b). By Euclid III.21, we have the angle β at point *D* as given (think of sliding point *B* around to point *D*, and we have angle γ at point *D* as given (thin of sliding point *C* around to point *D*). So we have $\delta = \beta + \gamma$. Euclid I.32 states that the sum of the angles of a triangle is equal to two right angles, so in triangle *ABC* we have $\alpha + \beta + \gamma = 2 ⦜$. Hence, $\alpha + \delta = 2 ⦜$, as claimed. □

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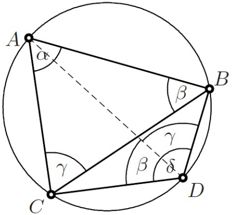

Figure 2.15(b)

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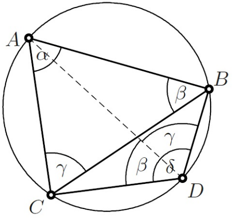

Figure 2.15(b)

Euclid, Book III Proposition 35

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Proof. Euclid's (lengthy) proof avoids the use of Thales' Theorem (Theorem 1.1). Instead, we claim that triangles AEC and DEB are similar, and then use Thale's Theorem. The opposite angles labeled ε in Figure 2.16(b) are equal by Euclid I.15. The angles at points A and D are equal (labeled α in Figure 2.16(b)) by Euclid III.21 (consider them as angles in triangles CAB and CDB with common base CB) and, similarly, the angles at points C and B are equal. So the AEC and DEB are similar, as claimed, and by Thale's Theorem $AE/CE = ED/EB$, or $AE \cdot EB = CE \cdot ED$, as claimed □

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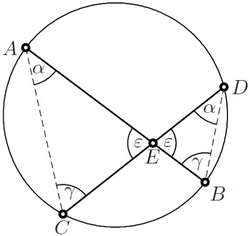

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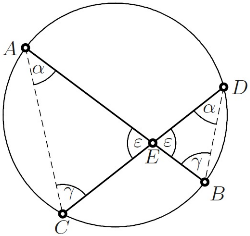

□

Figure 2.16(b)

Euclid, Book III Proposition 36

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Proof. The two angles labeled α in Figure 2.17(a) are equal by Euclid III.21 and Exercise 2.17. In triangle ATE the angle at A corresponds to the angle at T in triangle TBE, and both of these angles are α. In triangle ATE the angle at E corresponds to the angle at E in triangle TBE and this common angle is ε.

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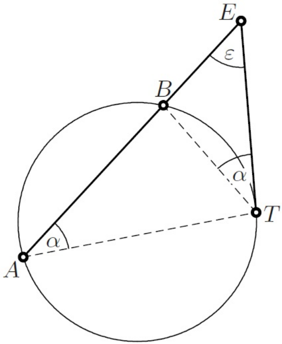

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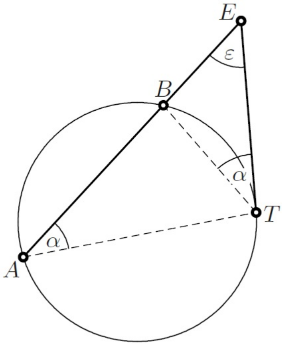

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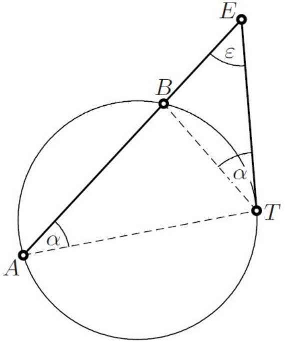

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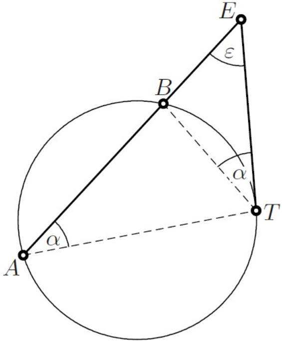

Figure 2.17(a)

Corollary (Clavius 1574)

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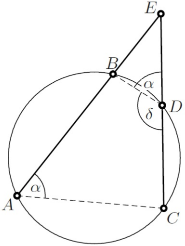

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Proof. By Euclid III.36, $AE \cdot BE = (TE)^2$ where T is the point of tangency of a tangent through E. Similarly, Euclid III.36 implies that $CE \cdot DE = (TE)^2$. Therefore, $AE \cdot BE = DE \cdot DE$, as claimed. □

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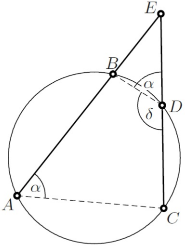

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