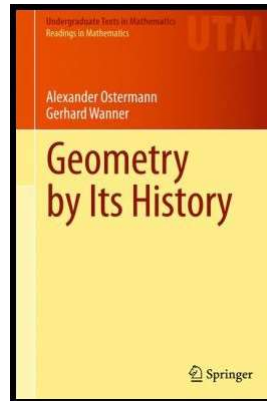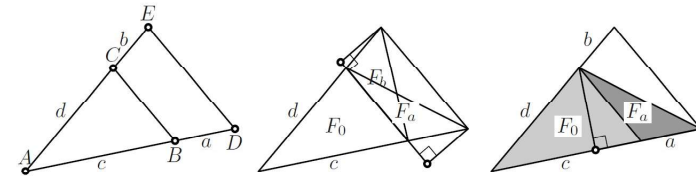# History of Geometry

**Chapter 2. The Elements of Euclid**

2.3. Books V and VI. Real Numbers and Thales' Theorem—Proofs of Theorems

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# Euclid, Book VI Proposition 2

**Euclid, Book VI Proposition 2.** Consider triangle $ADE$. Suppose $B$ is a point on segment $AD$ and $C$ is a point on segment $AE$ such that $BC$ is parallel to $DE$. Then $a/c = b/d$ (where $a, b, c, d$ are the distances given in the figure).



**Proof.** We start by joining points $B$ and $E$, and points $C$ and $D$. This gives two triangles, $CBE$ and $CBD$, with the same base, $CB$, and the same altitude (the distance from line $CB$ to point $E$ and to point $D$, but these are the same since lines $CB$ and $ED$ are parallel by hypothesis). Hence the two triangles have the same area, say $F_a = F_b$ (see the middle figure in which the altitudes are drawn, though they are not parts of the triangles).

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# Euclid, Book VI Proposition 2 (continued)

**Proof (continued).**
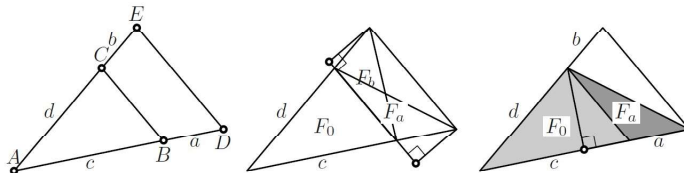


Thus if $F_0$ denotes the area of triangle $ABC$, then

$$F_a = F_b \text{ implies } \frac{F_a}{F_0} = \frac{F_b}{F_0} \implies \frac{a}{c} = \frac{b}{d}$$

since $F_a/F_0 = a/c$ by Euclid VI.1 (here we are using the "altitude on $AD$" as opposed to the altitude perpendicular to the base $CB$ [in the figure, right]; we claim that the ratios of the altitudes are the same), and similarly $F_b/F_0 = b/d$. $\qquad\square$

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# Euclid, Book VI Proposition 3

**Euclid, Book VI Proposition 3.** Consider triangle $ABC$ with angle $\gamma$ at point $C$. Let $CD$ be the bisector of $\gamma$. Then $a/b = p/q$ (where $a, b, p, q$ are the distances given in the figure).
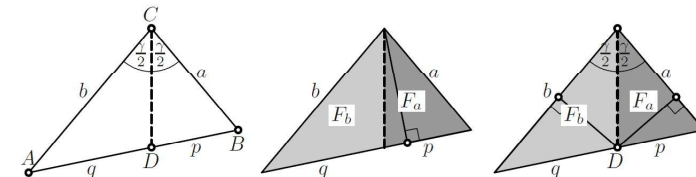


**Proof.** Let $F_a$ and $F_b$ be the areas of triangle $DBC$ and $ADC$, respectively. These have the same "altitude on $AB$" (see the middle figure). Hence $F_a/F_b = p/q$ by Euclid VI.1. Euclid I.26 implies that any point on the bisector of angle $\gamma$ is equidistant from the sides of the angle (as illustrated in the figure, right, for point $D$). So the altitude of triangle $ADC$ on $AC$ equals the altitude of triangle $DBC$ on $BC$, and by Euclid VI.1 we have $F_a/F_b = a/b$. Hence, $a/b = p/q$, as claimed. $\qquad\square$