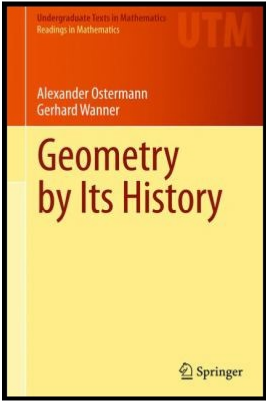
History of Geometry

Chapter 2. The Elements of Euclid

2.3. Books V and VI. Real Numbers and Thales' Theorem—Proofs of Theorems



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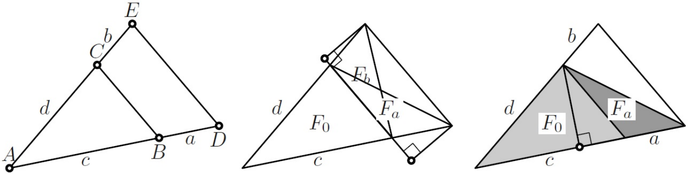
Euclid, Book VI Proposition 2

Euclid, Book VI Proposition 2. Consider triangle ADE. Suppose B is a point on segment AD and C is a point on segment AE such that BC is parallel to DE. Then $a/c = b/d$ (where a, b, c, d are the distances given in the figure).



Proof. We start by joining points B and E, and points C and D. This gives two triangles, CBE and CBD, with the same base, CB, and the same altitude (the distance from line CB to point E and to point D, but these are the same since lines CB and ED are parallel by hypothesis). Hence the two triangles have the same area, say $F_a = F_b$ (see the middle figure in which the altitudes are drawn, though they are not parts of the triangles).

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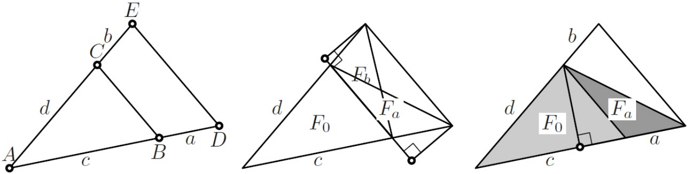

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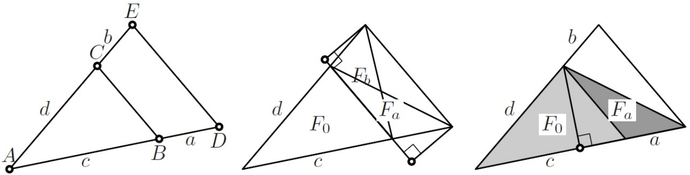
Euclid, Book VI Proposition 2 (continued)

Proof (continued).



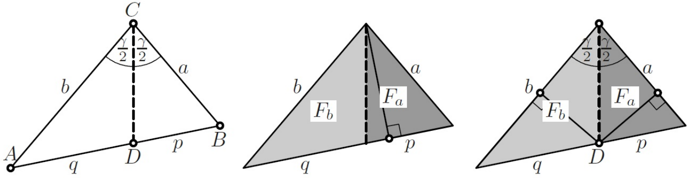
Thus if F_0 denotes the area of triangle ABC, then

$$F_a = F_b \text{ implies } \frac{F_a}{F_0} = \frac{F_b}{F_0} \implies \frac{a}{c} = \frac{b}{d}$$

since $F_a/F_0 = a/c$ by Euclid VI.1 (here we are using the "altitude on AD" as opposed to the altitude perpendicular to the base CB [in the figure, right]; we claim that the ratios of the altitudes are the same), and similarly $F_b/F_0 = b/d$. □

Euclid, Book VI Proposition 3

Euclid, Book VI Proposition 3. Consider triangle ABC with angle γ at point C. Let CD be the bisector of γ. Then $a/b = p/q$ (where a, b, p, q are the distances given in the figure).



Proof. Let F_a and F_b be the areas of triangle DBC and ADC, respectively. These have the same "altitude on AB" (see the middle figure). Hence $F_a/F_b = p/q$ by Euclid VI.1.

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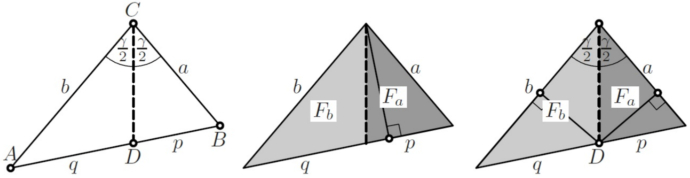

Proof. Let F_a and F_b be the areas of triangle DBC and ADC, respectively. These have the same "altitude on AB" (see the middle figure). Hence $F_a/F_b = p/q$ by Euclid VI.1. Euclid I.26 implies that any point on the bisector of angle γ is equidistant from the sides of the angle (as illustrated in the figure, right, for point D). So the altitude of triangle ADC on AC equals the altitude of triangle DBC on BC, and by Euclid VI.1 we have $F_a/F_b = a/b$. Hence, $a/b = p/q$, as claimed. □

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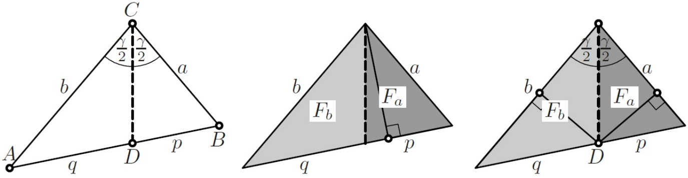

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