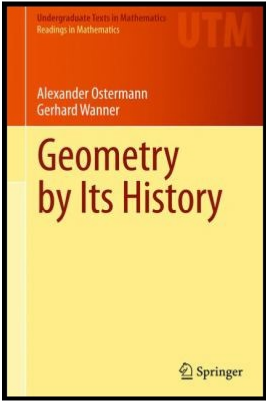
History of Geometry

Chapter 2. The Elements of Euclid

2.6. Book XII. Areas and Volumes of Circles, Pyramids, Cones—Proofs of Theorems



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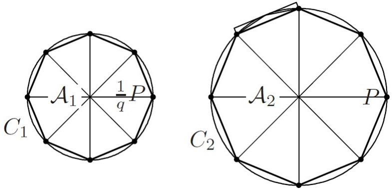

Figure 2.33

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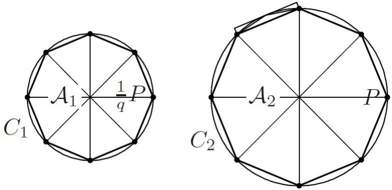

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Euclid, Book XII Proposition 2 (continued)

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Proof (continued). Next, divide the polygon P by q (that is, "shrink" its sides by a factor of q) to produce a new polygon A similar to P. By Euclid VI.9 the area of Q is $\mathcal{P}/q^2$, and since the radius of circle C_2 divided by q is the radius of C_1 then polygon Q can be inscribed in circle C_1. Then Euclid X.1 (again) gives $\mathcal{P}/q^2 < \mathcal{A}_1$, which implies $\mathcal{P} < q^2\mathcal{A}_1$, a CONTRADICTION to the above inequality. So the assumption is false, and we must have $\mathcal{A}_1/\mathcal{A}_2 \leq q^2$. Similarly, by interchanging the roles of C_1 and C_2 (and replacing the scaling factor with $1/q$) we can show that $\mathcal{A}_1/\mathcal{A}_2 < q^2$ is false and that $\mathcal{A}_1/\mathcal{A}_2 \geq q^2$. Therefore, $\mathcal{A}_1/\mathcal{A}_2 = q^2$, as claimed. □

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