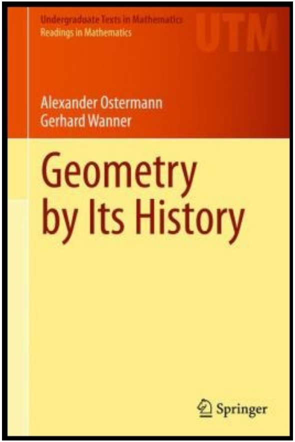
History of Geometry

Chapter 3. Conic Sections

3.1. The Parabola—Proofs of Theorems



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Theorem 3.1 (Apollonius' Proposition I.11)

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Proof. For a point P outside of a sphere, all line segments from P to the sphere which are tangent to the sphere are of the same length (we take this as true, and refer to Figure 3.2 left). These segments form a cone and intersect the sphere in a circle.

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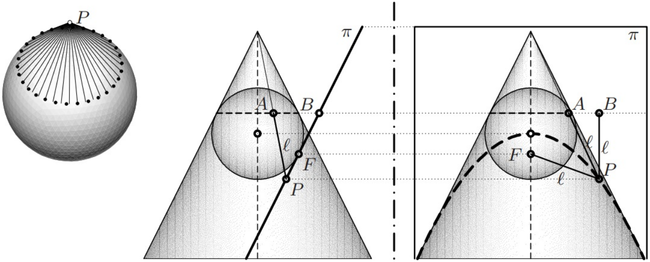

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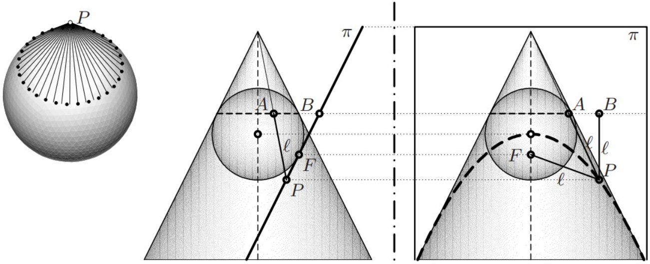

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Theorem 3.1 (Apollonius' Proposition I.11); Continued 1

Proof (continued). Next, introduce a Dandelin sphere in the cone which touches the cone in a circle C, as shown in Figure 3.2 (center). Then introduce a plane π that is parallel to a generator (that is, π is parallel to some ray which starts at vertex P and lies on the cone) and is tangent to the sphere at point F (alternatively, we could introduce plane π first and then the Dandelin sphere). Let P be an arbitrary point on the intersection of the cone with the plane. Let point A be the point on the intersection of the circle C with the generator of the cone that contains point P.

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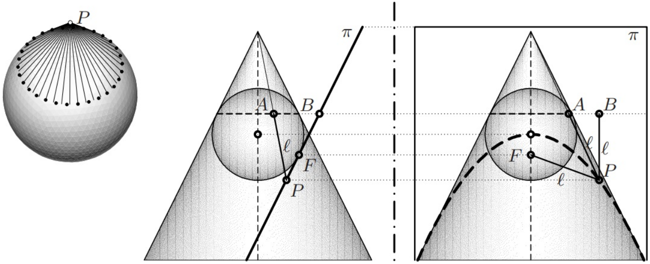

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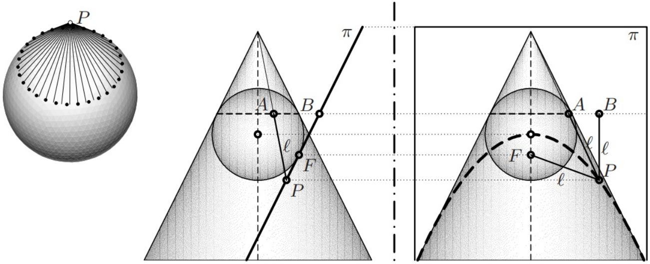

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Proof (continued). Now the plane containing the circle C intersects plane π in a line ℓ; we will see that this line is the directrix of the parabola. Let B be the point on the directrix above point P (so that segment BP is perpendicular to line ℓ). See Figure 3.2 (right). Then segment PA is tangent to the sphere (since it is part of a generator) and PF is tangent to the sphere since it lies in plane π and π is tangent to the sphere at point F (by construction). Hence, as observed above, this means that segments PA and PF are of the same length.

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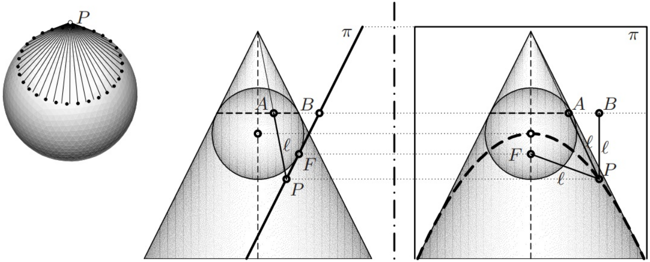

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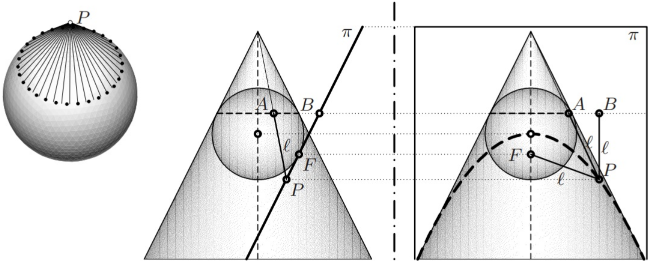

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Proof (continued). The slope of the line segment PF as given in Figure 3.2 (center) is the same as the slope of line segment PA when viewed in a vertical plane containing the generator containing PA (since plane π has the same "slope" as a generator of the cone, as Ostermann and Wanner put it; see page 63). Since the vertical displacement from P to A is the same as the vertical displacement from P to B, then by congruent triangles the length of PA equals the length of PB. Therefore the length of PF equals the length of PB and hence the distance from point P to the focus F is the same as the distance of P to the directrix. □

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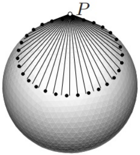

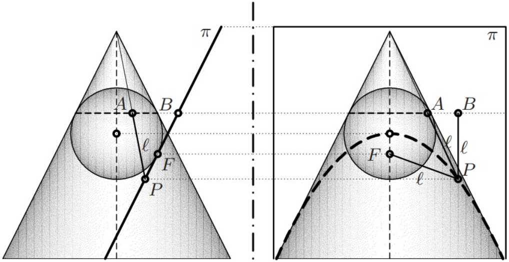

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