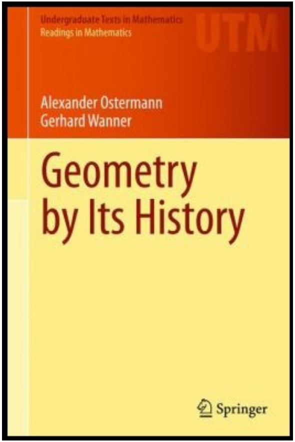
History of Geometry

Chapter 3. Conic Sections

3.2. The Ellipse—Proofs of Theorems



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Proof. Let π be a plane intersecting a cone and let π be less steep than the generators of the cone. Let P be an arbitrary point on the intersection of plane π and the cone. Next, introduce a Dandelin sphere in the cone which touches the cone in a circle C and is tangent to plane π at point F, as shown in Figure 3.3 (left).

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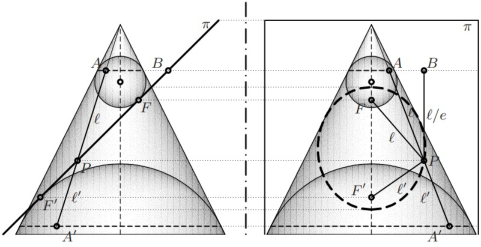

Fig. 3.3. An ellipse as the intersection of a cone with a plane

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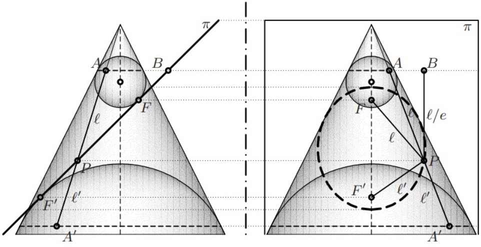

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Theorem 3.2.A (Apollonius' Proposition III.52); Cont. 1

Proof (continued). Now the plane containing circle C intersects plane π in a line. Let B be the point on this line that is above point P (see Figure 3.3, right). Since π is less steep than the generator $\overleftrightarrow{AP}$ of the cone, then PB is longer than PA (compare the slopes of these line segments in the plane containing points P, A, and B). Define the factor by which PB is longer than PA as $1/e$ where $0 < e < 1$.

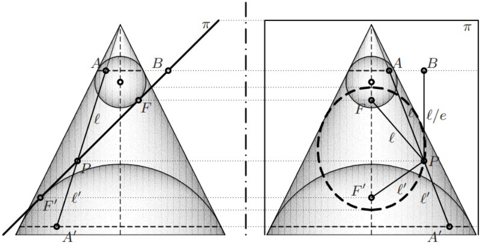


Fig. 3.3. An ellipse as the intersection of a cone with a plane

Theorem 3.2.A (Apollonius' Proposition III.52); Cont. 2

Proof (continued). Since parameter $1/e$ is determined by the slope of a generator of the cone and the slope of plane π (more appropriately, $1/e$ is determined by the slopes of the *cross sections* of the cone and π in the plane containing points P, A, and B), then $1/e$ is independent of point P. Notice that both PF and PA are tangent to the Dandelin sphere, so they have the same lengths (as argued at the beginning of the proof of Theorem 3.1).

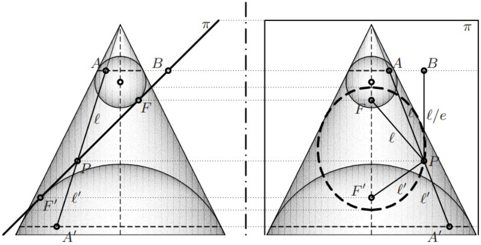


Fig. 3.3. An ellipse as the intersection of a cone with a plane

Theorem 3.2.A (Apollonius' Proposition III.52); Cont. 3

Proof (continued). Next, place a second (larger) Dandelin sphere below plane π that intersects the cone in a circle C' and is tangent to π at a point F' (see Figure 3.3, left). Extend AP so that it intersects circle C' at P'. Again, PF' and PA' are the same length. Similar to the argument above, we have that PB' is longer than PA' by the factor $1/e$ (where B' is the point of intersection of $\overleftrightarrow{BP}$ with the plane containing C'; point B' is not in Figure 3.3).

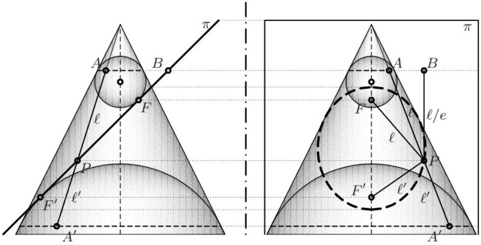
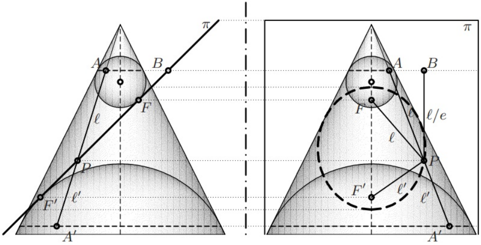


Fig. 3.3. An ellipse as the intersection of a cone with a plane

Theorem 3.2.A (Apollonius' Proposition III.52); Cont. 4

Proof (continued). Since circles C and C' lie in paralel planes, then the sum of the length of PA (denoted ℓ in Figure 3.3) and the length of PA' (denoted ℓ') is constant (namely $\ell + \ell'$). Since the length of PF is ℓ and the length of PF' is ℓ, then for each point P on the ellipse the sum of the distance of P from F and F' is constant, as claimed. □



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