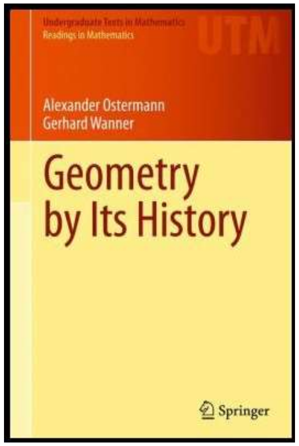
History of Geometry

Chapter 3. Conic Sections

3.3. The Hyperbola—Proofs of Theorems



Theorem 3.3.A (Apollonius' Proposition III.51)

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The intersection of a double cone and a plane that is more steep than the generators of the cone is a locus of all points in a plane whose distances from two fixed points in the plane (called foci) have a constant difference.

Proof. Let π be a plane intersecting a double cone and let π be more steep than the generators of the double cone. Let P be an arbitrary point on the intersection of plane π and the lower cone (a similar argument holds when P is on the upper cone).

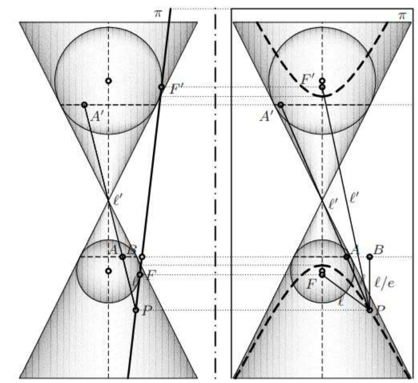

Fig. 3.10. A hyperbola as the intersection of a cone with a plane

Theorem 3.3.A (Apollonius' Proposition III.51); Cont. 1

Proof (continued). Next, introduce two Dandelin spheres, one in the lower cone tangent to π and one in the upper cone tangent of π, and let F and F' be the points of tangency of the upper and lower Dandelin cones with plane π, respectively. Let circle C be the intersection of the lower cone and π and circle C' be the intersection of the lower cone and π and circle C' be the intersection of the upper cone and π. See Figure 3.10 (left).

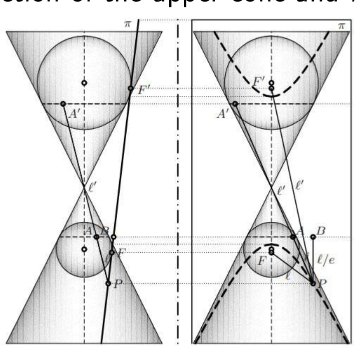

Fig. 3.10. A hyperbola as the intersection of a cone with a plane

Theorem 3.3.A (Apollonius' Proposition III.51); Cont. 2

Proof (continued). Now the plane containing circle C intersects plane π in a line. Let B be the point on this line that is above point P (see Figure 3.10, right). Since π is less steep than the generator $\overleftrightarrow{AP}$ of the cone, then PB is longer than PA (compare the slopes of these line segments in the plane containing points P, A, and B). Define the factor by which PB is longer than PA as $1/e$ where $e > 1$.

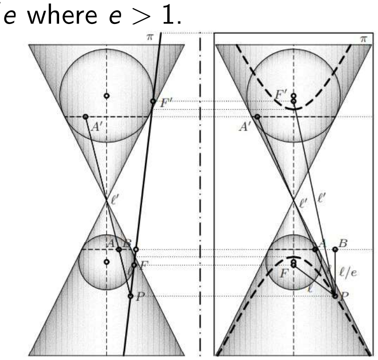

Fig. 3.10. A hyperbola as the intersection of a cone with a plane

Theorem 3.3.A (Apollonius' Proposition III.51); Cont. 3

Proof (continued). Since parameter $1/e$ is determined by the slope of a generator of the cone and the slope of plane π (more appropriately, $1/e$ is determined by the slopes of the *cross sections* of the cone and π in the plane containing points P, A, and B), then $1/e$ is independent of point P. Notice that both PF and PA are tangent to the Dandelin sphere, so they have the same lengths (as argued at the beginning of the proof of Theorem 3.1).

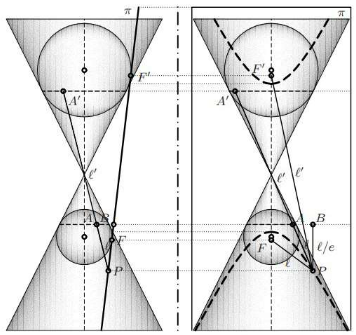

Fig. 3.10. A hyperbola as the intersection of a cone with a plane

Theorem 3.3.A (Apollonius' Proposition III.51); Cont. 4

Proof (continued). Extend AP through the vertex of the double cone and then to the second cone and circle C' at point A'. Again, PF' and PA' are the same length. Next, the plane containing circle C' intersects plane π is a line. Let B' be the point on this line that is above P (s that PB and PB' are collinear; point B' is not in Figure 3.10). Just as PB is shorter than PA above, PB' is shorter than PA' by a factor of $1/e$ and parameter q/e is independent of P.

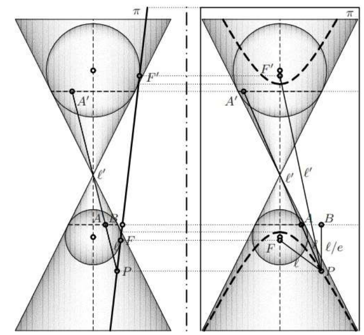

Fig. 3.10. A hyperbola as the intersection of a cone with a plane

Theorem 3.3.A (Apollonius' Proposition III.51); Cont. 5

Proof (continued). With ℓ as the length of of PF (and PA) then PB is length ℓ/e. With ℓ' as the length of PF' (and PA') then PB' is length ℓ'/e. For any point P on the lower branch of the hyperbola, we have that the length of PF' minus the length of PF is $\ell' = \ell$. Also, for any point on the lower branch, $\ell'/e - \ell/e$ is the same (since PB and PB' are collinear). Therefore, $\ell' - \ell$ is the same for all points P on the lower branch of the hyperbola. That is the distances of P from two fixed points in the plane have a constant difference, as claimed. □

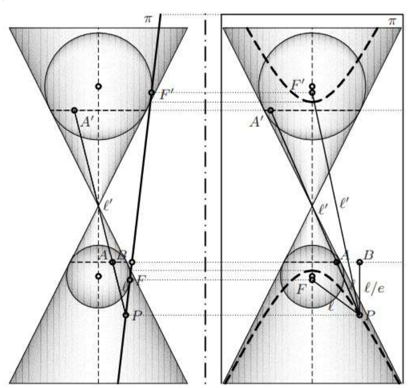

Fig. 3.10. A hyperbola as the intersection of a cone with a plane