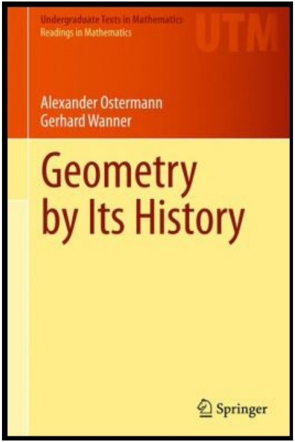
History of Geometry

Chapter 3. Conic Sections

3.4. The Area of a Parabola—Proofs of Theorems



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Theorem 3.4.A (Archimedes)

Theorem 3.4.A. With $\mathcal{P}$ as the area under the parabola given in Figure 3.12 (left) and with $\mathcal{T}$ as the area of the large isosceles triangle, we have $\mathcal{P} = \frac{4}{3}\mathcal{T}$.

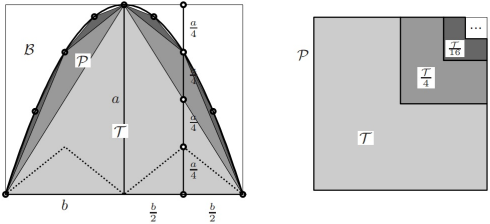

Fig. 3.12. The quadrature of the parabola

Proof. Let the base of the large light-grey isosceles triangle be $2b$ and the height be a. In terms of coordinates, we have the point (b, a) on the parabola $y = x^2$ so that $a = b^2$ (this is where we use the fact that the curve is a parabola; of course, we could scale the y-coordinate to deal with a more general case). By hypothesis, the area of this triangle is $\mathcal{T}$. Then the area is $\mathcal{T} = ab$.

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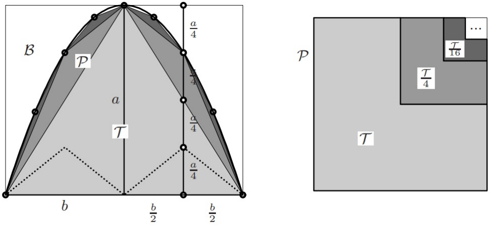

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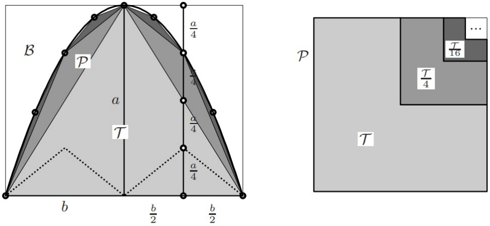
Theorem 3.4.A (continued 1)

Proof (continued).



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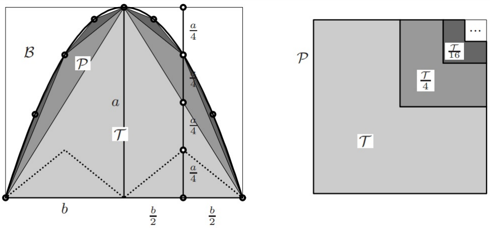
Next, we bisect the right half of the base of the light-grey triangle and introduce a line segment perpendicular to the base. We see that this results in the point $(b/2, a/4)$ on the parabola $y = x^2$, since $a/4 = (b/2)^2$ (because $b^2 = a$). So the medium-grey triangle on the right has base b and height $a/4$, and therefore area $ab/8$. There is a second medium-grey triangle on the left of the same dimensions, so the area of the two medium-grey triangles together is $ab/4 = \mathcal{T}/4$.

Theorem 3.4.A (continued 2)

Proof (continued).



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Similarly by bisecting, the four parts of the base we get four dark-grey triangles, each of area $ab/64 = \mathcal{T}/64$. Summing we get a total dark-grey area of $\mathcal{T}/16$ Recursively, for each natural number n we get 2^{n-1} triangles of total area $\mathcal{T}/4^{n-1}$. We can now sum a series to get

$$\sum_{n=1}^{\infty} \mathcal{T}/4^{n-1} = 4\mathcal{T}\frac{(1/4)}{1-(1/4)} = \frac{4}{3}\mathcal{T},$$

as claimed.

Theorem 3.4.A (continued 2)

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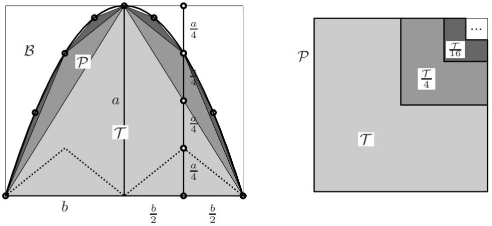

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Theorem 3.4.A (continued 3)

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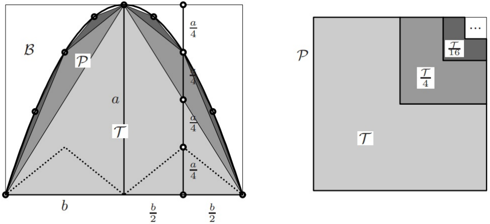

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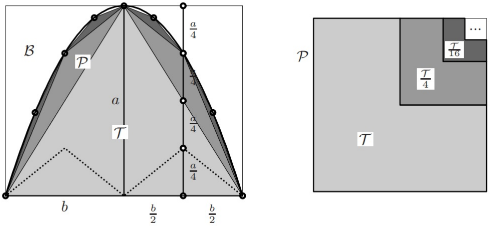
Alternatively (if we want to avoid the use of infinite series), we can take $\mathcal{T} = ab$ and partition it into three equal squares, each of area $ab/3$ (and sides of length $\sqrt{ab/3}$), and arrange them as given in Figure 3.12 (right) in light-grey. We similarly partition the medium-grey area $\mathcal{T}/4$ into three equal squares, each of area $ab/12$ (and sides of length $\sqrt{ab/12} = \sqrt{ab/3}/2$), and arrange them as given in Figure 3.12 (right) in medium-grey. Recursively we can arrange the other areas $\mathcal{T}/4^{n-1}$ similarly and see that the resulting total area is $4\mathcal{T}/3$ (by comparing $\mathcal{T}$ to the total area in Figure 3.12 right), as claimed. □

Theorem 3.4.A (continued 3)

Proof (continued).



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