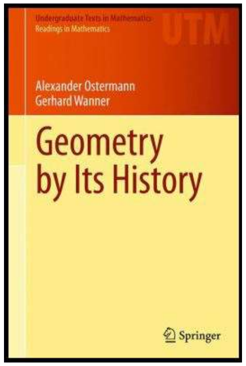
History of Geometry

Chapter 4. Further Results in Euclidean Geometry

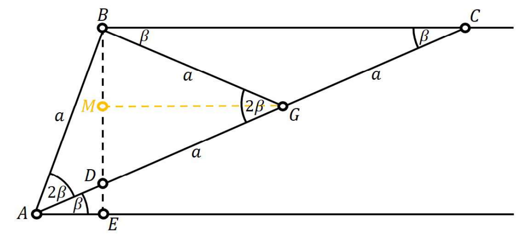
4.1. The Conchoid of Nicodemes, the Trisection of an Angle—Proofs of Theorems



Theorem 4.1.A. Pappus' Proposition IV.32 in *Collection*

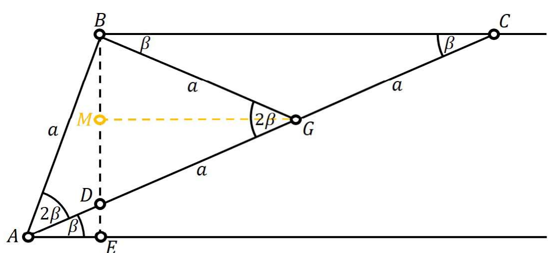
Theorem 4.1.A. (Pappus' Proposition IV.32 in *Collection*)
Assuming the existence of the conchoid of Nicomedes, we can trisect any angle.

Proof. Let β be the angle EAD in Figure 4.3 (modified; see below). Let G be the midpoint of segment DC, so that $DG = GC = a$. Construct a parallel to BC through point G (Euclid, Book I Proposition 31), and let M be the point of intersection of this line with BD. By Euclid, Book I Proposition 29, angle CBD (a right angle) equals angle GMD, and angle MGD equals angle BCD.



Theorem 4.1.A (continued)

Proof (continued).

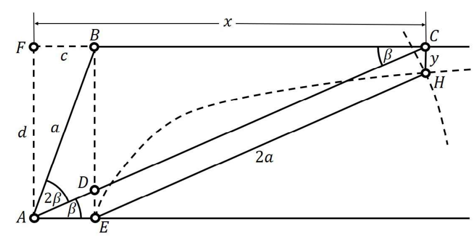


So triangle CBD is similar to triangle GMD since the corresponding angles are equal (Euclid, Book VI Proposition 4), and since $DG = \frac{1}{2}DC$ then $DM = \frac{1}{2}BD$. Triangle BMG and triangle DMG are congruent by SAS (Euclid, Book I Proposition 4) so that $BG = a$ and triangles BGC and GBD are isosceles. Consequently, we have β at C (alternate angles, Euclid Book I Proposition 29), β at B (isosceles triangle, Euclid Book I Proposition 5), 2β at G (exterior angle, Euclid Book I Proposition 32), and angle BAG is 2β (isosceles triangle), so β is 1/3 of the original angle α at A, as claimed. □

Theorem 4.1.B. Pappus' Proposition IV.31

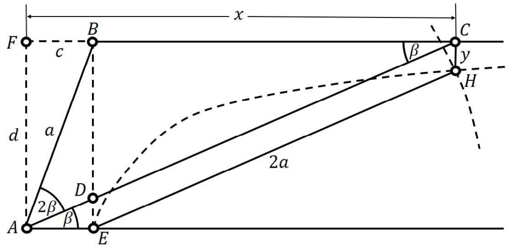
Theorem 4.1.B. (Pappus' Proposition IV.31 in *Collection*)
Assuming the existence of the hyperbola, we can trisect any angle.

Proof. For the argument, we borrow some results from analytic geometry by introducing a coordinate system. Let point F be the origin of a Cartesian coordinate system with x-axis running horizontally and y-axis running vertically (but with the positive direction as downward; then and (x, y)-coordinate system is left-handed, or we could deal with a right-handed (y, x)-coordinate system, but neither of these choices will affect our proof). See the modified version of Figure 4.3 below.



Theorem 4.1.B (continued 1)

Proof (continued).

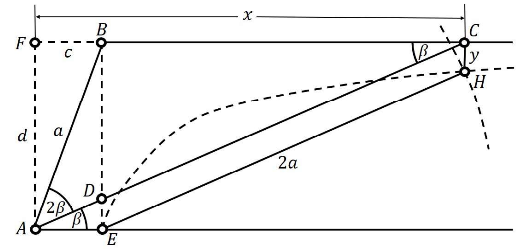


Introduce the hyperbola $xy = cd$. This has asymptotes of FA (the "y-axis") and FB (the "x-axis"), and passes through point E which has coordinates $x = c$ and $y = d$. Take a circle of radius $2a$ with center E and find its intersection with the hyperbola at point H (which we give as having coordinates x and y, fixed values here). Construct a perpendicular to BC through point H (Euclid, Book I Proposition 11) and let the intersection of BC and the perpendicular be point C. Construct line AC (Euclid, Postulate 1) and let D be the point of intersection of AC with BE.

Theorem 4.1.B. Pappus' Proposition IV.31 in *Collection*

Theorem 4.1.B (continued 2)

Proof (continued).



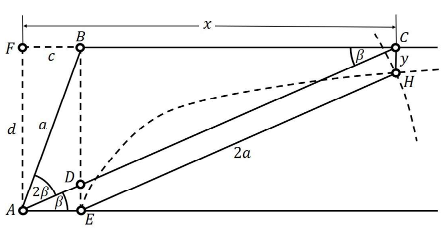
Angles FCA and DAE are equal (Euclid, Book I Proposition 29), say they have measure β. Since angles AFC and AED are right angles by construction, then angles FAC and ADE are equal (Euclid, Book I Proposition 32). Hence, triangles AFC and DEA are similar (Euclid, Book VI Proposition 4) and so $\frac{DE}{AF} = \frac{DE}{d} = \frac{AE}{CF} = \frac{c}{x}$ or $x(DE) = cd$. But we have for these distances that $xy = cd$, so we must have $DE = y$. Then by Euclid, Book I Proposition 33, DC is parallel to EH. So $CDEH$ is a parallelogram and $CD = 2a$ (Euclid, Book I Proposition 34).

Theorem 4.1.B. Pappus' Proposition IV.31 in *Collection*

Theorem 4.1.B (continued 3)

Theorem 4.1.B. (Pappus' Proposition IV.31 in *Collection*)
Assuming the existence of the hyperbola, we can trisect any angle.

Proof (continued).



So we have that line AC satisfies the condition that it intersects BE at point D and the distance $DC = 2a$. As argued in Theorem 4.1.A, this implies that $\beta = \alpha/3$. □

NOTE. We found the same point C as in Theorem 4.1.A, but we did so using the hyperbola, and avoided the use of the conchoid in this proof.