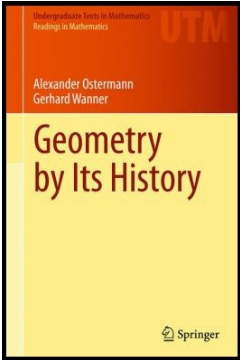
History of Geometry

Chapter 4. Further Results in Euclidean Geometry

4.1. The Conchoid of Nicodemes, the Trisection of an Angle—Proofs of Theorems



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Theorem 4.1.A. Pappus' Proposition IV.32 in *Collection*

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Proof. Let β be the angle EAD in Figure 4.3 (modified; see below). Let G be the midpoint of segment DC, so that $DG = GC = a$. Construct a parallel to BC through point G (Euclid, Book I Proposition 31), and let M be the point of intersection of this line with BD. By Euclid, Book I Proposition 29, angle CBD (a right angle) equals angle GMD, and angle MGD equals angle BCD.

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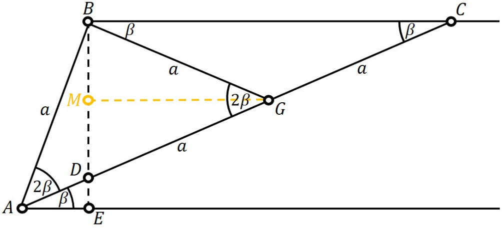

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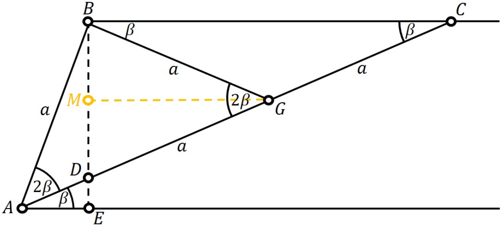
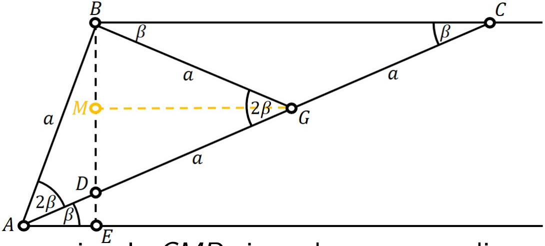


Theorem 4.1.A (continued)

Proof (continued).



So triangle CBD is similar to triangle GMD since the corresponding angles are equal (Euclid, Book VI Proposition 4), and since $DG = \frac{1}{2}DC$ then $DM = \frac{1}{2}BD$. Triangle BMG and triangle DMG are congruent by SAS (Euclid, Book I Proposition 4) so that $BG = a$ and triangles BGC and GBD are isosceles. Consequently, we have β at C (alternate angles, Euclid Book I Proposition 29), β at B (isosceles triangle, Euclid Book I Proposition 5), 2β at G (exterior angle, Euclid Book I Proposition 32), and angle BAG is 2β (isosceles triangle), so β is 1/3 of the original angle α at A, as claimed. □

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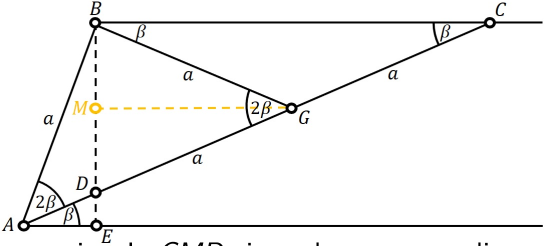

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Theorem 4.1.B. Pappus' Proposition IV.31

Theorem 4.1.B. (Pappus' Proposition IV.31 in *Collection*)
Assuming the existence of the hyperbola, we can trisect any angle.

Proof. For the argument, we borrow some results from analytic geometry by introducing a coordinate system. Let point F be the origin of a Cartesian coordinate system with x-axis running horizontally and y-axis running vertically (but with the positive direction as downward; then and (x, y)-coordinate system is left-handed, or we could deal with a right-handed (y, x)-coordinate system, but neither of these choices will affect our proof). See the modified version of Figure 4.3 below.

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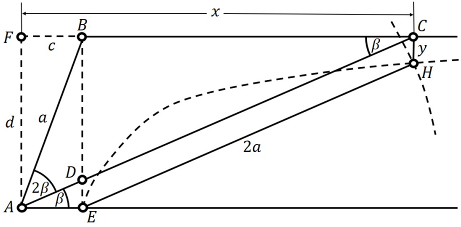

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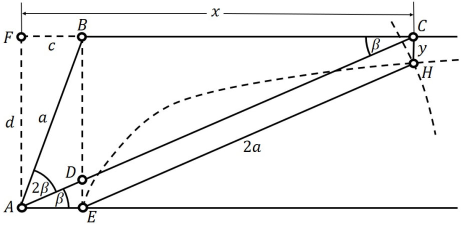
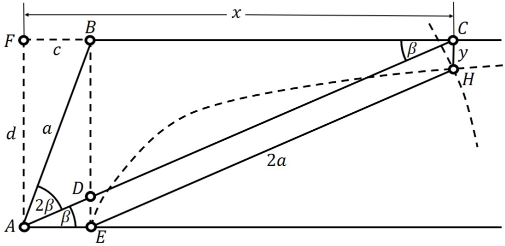


Theorem 4.1.B (continued 1)

Proof (continued).



Introduce the hyperbola $xy = cd$. This has asymptotes of FA (the "y-axis") and FB (the "x-axis"), and passes through point E which has coordinates $x = c$ and $y = d$. Take a circle of radius $2a$ with center E and find its intersection with the hyperbola at point H (which we give as having coordinates x and y, fixed values here). Construct a perpendicular to BC through point H (Euclid, Book I Proposition 11) and let the intersection of BC and the perpendicular be point C. Construct line AC (Euclid, Postulate 1) and let D be the point of intersection of AC with BE.

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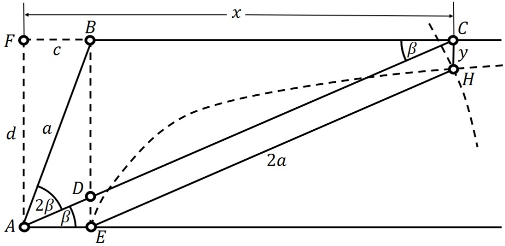

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Theorem 4.1.B (continued 2)

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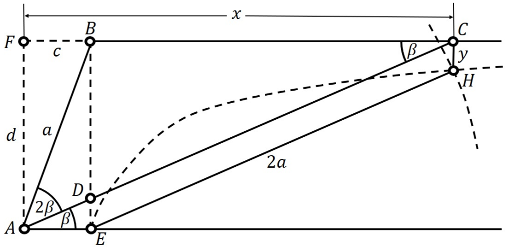

Angles FCA and DAE are equal (Euclid, Book I Proposition 29), say they have measure β. Since angles AFC and AED are right angles by construction, then angles FAC and ADE are equal (Euclid, Book I Proposition 32). Hence, triangles AFC and DEA are similar (Euclid, Book VI Proposition 4) and so $\frac{DE}{AF} = \frac{DE}{d} = \frac{AE}{CF} = \frac{c}{x}$ or $x(DE) = cd$. But we have for these distances that $xy = cd$, so we must have $DE = y$. Then by Euclid, Book I Proposition 33, DC is parallel to EH. So $CDEH$ is a parallelogram and $CD = 2a$ (Euclid, Book I Proposition 34).

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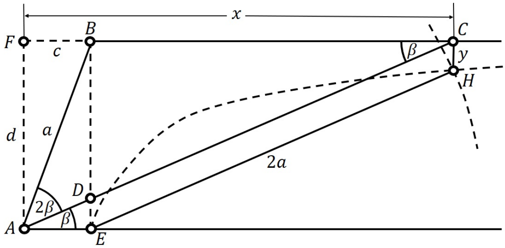

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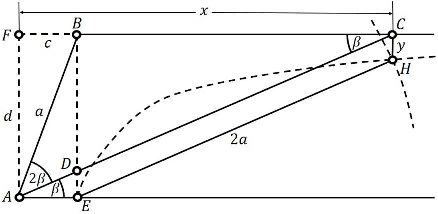

So we have that line AC satisfies the condition that it intersects BE at point D and the distance $DC = 2a$. As argued in Theorem 4.1.A, this implies that $\beta = \alpha/3$. □

NOTE. We found the same point C as in Theorem 4.1.A, but we did so using the hyperbola, and avoided the use of the conchoid in this proof.

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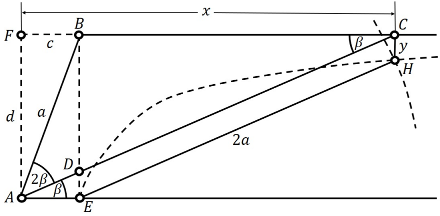

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