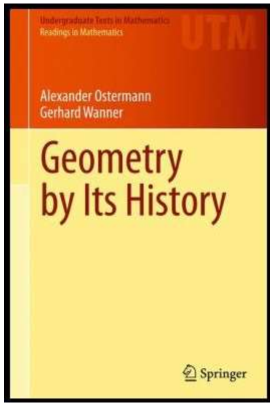
History of Geometry

Chapter 4. Further Results in Euclidean Geometry

4.2. The Archimedean Spiral—Proofs of Theorems



Proposition XXIV of *On Spirals*

Proposition XXIV. The area bounded by the first turn of the spiral and the initial line is equal to one-third of the 'first circle' [$= \frac{1}{3}\pi(2\pi a)^2$, where the spiral is $r = a\theta$].

Proof. Recall from Calculus 3 (MATH 2110) Section 11.5. Areas and Lengths in Polar Coordinates, if $r = f(\theta)$ in polar coordinates (r, θ), the the area bounded by $r = f(\theta)$ for $\theta = \alpha$ to $\theta = \beta$ is $A = \int_\alpha^\beta \frac{1}{2}(f(\theta))^2\, d\theta$. For the Archimedean spiral we have $r = f(\theta) = a\theta$, so in "the first turn of the spiral," the area bounded is

$$A = \int_0^{2\pi} \frac{1}{2}(a\theta)^2\, d\theta = \frac{a^2}{2}\int_0^{2\pi} \theta^2\, d\theta = \frac{a^2}{2}\frac{\theta^3}{3}\bigg|_0^{2\pi} = \frac{a^2}{2}\frac{(2\pi)^3}{3} - \frac{a^2}{2}\frac{(0)^3}{3}$$

$$= \frac{1}{3}a^2\frac{(2\pi)(2\pi)^2}{2} = \frac{1}{3}a^2\pi(2\pi)^2 = \frac{1}{3}\pi(2\pi a)^2,$$

as claimed. □