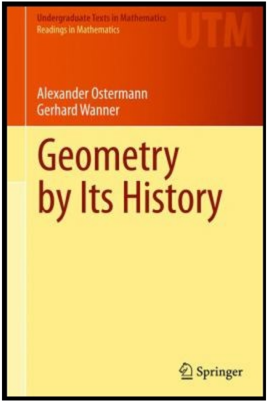
History of Geometry

Chapter 4. Further Results in Euclidean Geometry

4.2. The Archimedean Spiral—Proofs of Theorems



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Proposition XXIV of *On Spirals*

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Proof. Recall from Calculus 3 (MATH 2110) Section 11.5. Areas and Lengths in Polar Coordinates, if $r = f(\theta)$ in polar coordinates (r, θ), the the area bounded by $r = f(\theta)$ for $\theta = \alpha$ to $\theta = \beta$ is $A = \int_\alpha^\beta \frac{1}{2}(f(\theta))^2 \, d\theta$.

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For the Archimedean spiral we have $r = f(\theta) = a\theta$, so in "the first turn of the spiral," the area bounded is

$$A = \int_0^{2\pi} \frac{1}{2}(a\theta)^2\,d\theta = \frac{a^2}{2}\int_0^{2\pi} \theta^2\,d\theta = \frac{a^2}{2}\frac{\theta^3}{3}\bigg|_0^{2\pi} = \frac{a^2}{2}\frac{(2\pi)^3}{3} - \frac{a^2}{2}\frac{(0)^3}{3}$$

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