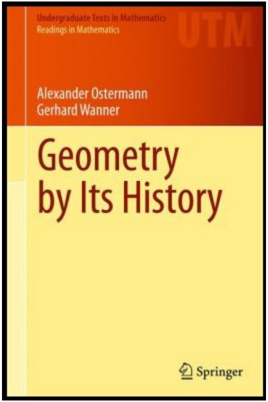
History of Geometry

Chapter 5. Trigonometry

5.1. Ptolemy and the Chord Function—Proofs of Theorems



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Lemma 5.1. Ptolemy's Theorem

Lemma 5.1 (Ptolemy's Theorem). Let a quadrilateral with sides a, b, c, d be inscribed in a circle. Then the diagonals δ_1 and δ_2 satisfy $ac + bd = \delta_1\delta_2$.

Proof. First, let E be the unique point on line segment AC such that the angle $\angle EDA$ equals in measure the angle $\angle CDB$ (the measures of these angles are labeled α in Figure 5.4).
Euclid's Proposition III.21 states:
"In a circle the angles in the same segment equal one another." This means that in a circle, if two angles inscribed in a circle (such an angle has its vertex and two points on the sides of the angle all on the circle) determine the same length chord, then the angles are equal in measure.

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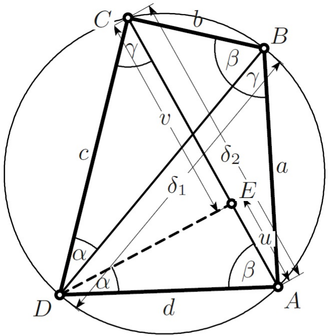

Fig. 5.4. Proof of Ptolemy's lemma

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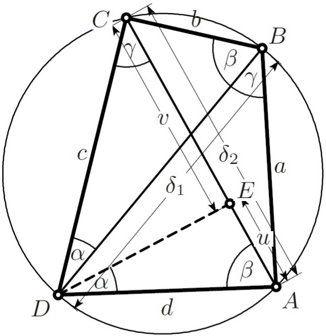

Fig. 5.4. Proof of Ptolemy's lemma

Lemma 5.1. Ptolemy's Theorem (continued)

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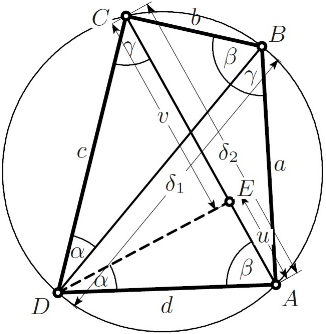
Proof. Since angles $\angle CBD$ and $\angle CAD$ both determine chord CD, then these angles are equal by Euclid's Proposition (the measures of these angles are labeled β in Figure 5.4). Therefore triangles $\triangle EDA$ and $\triangle CDB$ are similar (AAA) and so $b/\delta_1 = u/d$ and $a/\delta_1 = v/c$. This implies $bd + ac = \delta_1 u + \delta_1 v = \delta_1(u + v) = \delta_2\delta_2$, as claimed. □



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