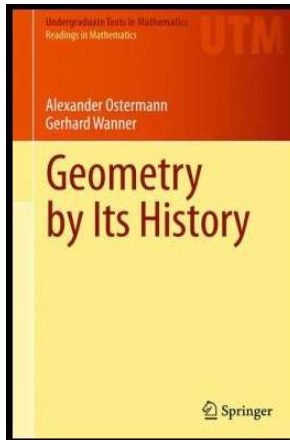


# History of Geometry

## Chapter 5. Trigonometry

### 5.10. The Great Discoveries of Kepler and Newton—Proofs of Theorems



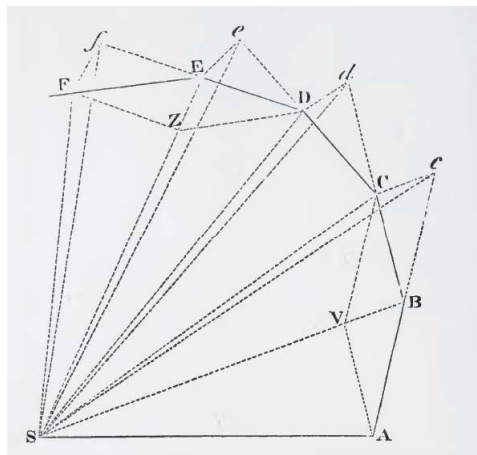
## Theorem 5.8 (Newton's Theorem 1)

**Theorem 5.8.** (Theorem 1 of Newton's *Principia Mathematica*.)  
“The areas, which revolving bodies describe by radii drawn to an immovable centre of force, do lie in some immovable planes, and are proportional to the times in which they are described.”

**Proof.** We go through the argument given in Ostermann and Wanner (which is the same as given by Newton in *Principia*). The argument is a discretized version of the physical problem. Newton appeals to another result (Lemma III in his Book I) to move from the discrete to the continuous case. We place the Sun at point  $S$  (the “centre of force”) and approximate the orbit with points  $A, B, C, D, E$ , etc. The force  $f$  then acts on the object orbiting the Sun by pulling the object in the direction of the Sun. We let the force act on the object at point  $A$  and then assume the object moves in a straight line over time period  $\Delta t$  without the force acting from  $A$  to  $B$  (the motion is along a straight line by Newton's First Law, or “Lex 1”). See the figure below.

## Theorem 5.8 (Newton's Theorem 1, continued 1)

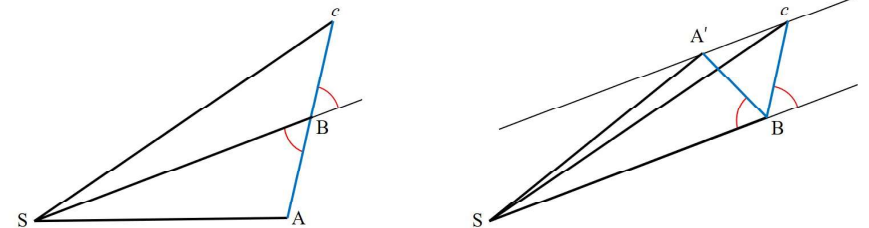
**Proof (continued).** Next, we let the force act again at point  $B$  by an amount  $f \Delta t$ . Now consider the triangles  $ABS$  and  $BcS$  in the figure. We claim that these triangles have the same areas. We argue this below.



The Figure for Newton's Theorem 1 from [Motte's 1846 translation](#) of *Principia*.

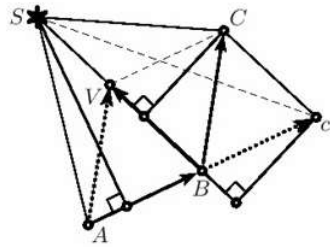
## Theorem 5.8 (Newton's Theorem 1, continued 2)

**Proof (continued).** To see that triangles  $ABS$  and  $BcS$  have the same area, reflect triangle  $ABS$  about the line containing points  $S$  and  $B$  and let  $A'$  denote the image of  $A$ . We then see that the triangles have a common base  $SB$  and the same altitude. So the areas are the same (by Euclid I.41).



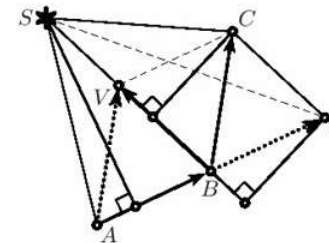
## Theorem 5.8 (Newton's Theorem 1, continued 3)

**Proof (continued).** Next, we let the force affect the object at point  $B$ . By Newton's Second Law (or Lex 2), the force acts along the line from  $B$  to  $S$ ; the "change of motion" (as it's called in Lex 2) is represented as the vector from  $B$  to  $V$  (we denote this vector, and similar vectors, as  $\overrightarrow{BV}$ ) in Figure 5.29 (right) below. Adding this change in motion to the velocity of the object before the application of the force (represented by the vector  $\overrightarrow{AB}$  in the figure; think of it as the change in position over time  $\Delta t$ ), we get the new velocity by adding vectors  $\overrightarrow{AB}$  and  $\overrightarrow{BV}$  to get the resultant vector  $\overrightarrow{AV}$ . We translate  $\overrightarrow{AV}$  to point  $B$  to get velocity vector  $\overrightarrow{BC}$ .



## Theorem 5.8 (Newton's Theorem 1, continued 4)

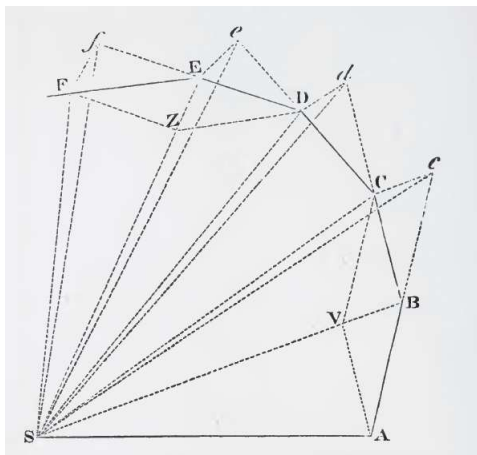
**Proof (continued).** By construction above, vector  $\overrightarrow{AB}$  equals vector  $\overrightarrow{BC}$ . So when translating vectors  $\overrightarrow{AB}$  and  $\overrightarrow{AV}$  to point  $B$ , we see that segment  $Cc$  is parallel to segment  $SB$ . So the area of triangles  $BCS$  and  $BcS$  are the same since they have the same base,  $SB$ , and the same heights (represented, respectively, in the figure by the segments with the right-angle symbols which end at  $C$  and  $c$ ; we are again using Euclid I.41 here).



## Theorem 5.8 (Newton's Theorem 1, continued 5)

**Proof (continued).** Since, as shown above, the triangles  $ABS$  and  $BcS$  have the same areas, then we have that triangles  $ABS$  and  $BCS$  have the same areas. Similar, the areas of triangles  $ABS$ ,  $BCS$ ,  $CDS$ ,  $DES$ , etc. are all the same.

So the claim holds when we have the discrete version of the problem with the force applied as impulses at equal time steps  $\Delta t$ .



The Figure for Newton's Theorem 1 from Motte's 1846 translation of *Principia*.

## Theorem 5.8 (Newton's Theorem 1, continued 6)

**Theorem 5.8.** (Theorem 1 of Newton's *Principia Mathematica*). "The areas, which revolving bodies describe by radii drawn to an immoveable centre of force, do lie in some immoveable planes, and are proportional to the times in which they are described."

**Proof (continued).** To complete the proof, Newton now applies his Lemma III which, in essence, involve taking a limit as  $\Delta t \rightarrow 0$ . Ostermann and Wanner describe Newton's approach as similar to the numerical technique called the "Euler method" which is used to approximate solutions to differential equations.

In the spirit of Calculus 1 (MATH 1910), we could consider the area swept out by the line segment joining the Sun and the orbiting object as a function time  $A(t)$ . We then have from Newton's discrete argument that  $\Delta A/\Delta t$  is a constant. So we have  $\lim_{\Delta t \rightarrow 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt}$  is constant (this is the constant of proportionality in Newton's theorem).  $\square$

# Newton's Lemma

**Newton's Lemma.** Let  $APQ$  be an ellipse with focus  $S$  and suppose  $P$  to be the position of the planet moving towards  $Q$ , while the point  $R$  moves on the tangent with  $S, Q, P$  collinear. Let  $T$  be the orthogonal projection of  $Q$  onto  $PS$  (see Figure 5.30, right). Then if the distance  $PQ$  tends to zero, we have  $RQ \approx (\text{Constant}) \cdot QT^2$ , where the constant is independent of the position of  $P$  on the ellipse.

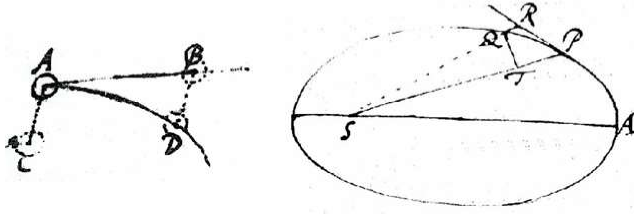


Fig. 5.30. Reproductions from Newton's autograph (1684), manuscript Cambridge Univ. Lib. Add. 3965<sup>6</sup>; the force acting on a moving body (left); picture for Newton's lemma (right). Reproduced by kind permission of the Syndics of Cambridge University Library

# Newton's Lemma, continued 1

**Proof.**

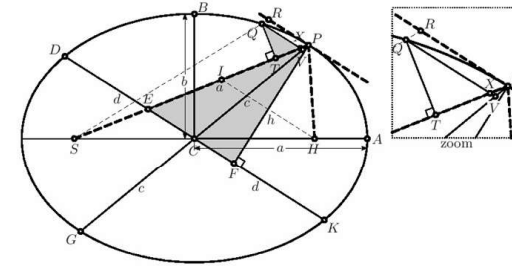


Fig. 5.31. Newton's proof of his lemma

From this figure, we have by Apollonius II.6 (see [Section 3.2. The Ellipse](#), Figure 3.7(b)) that the tangent  $PR$  is parallel to the "diameter"  $DCK$  which is conjugate to diameter  $GCP$ . Let the lengths of these diameters be  $2d$  and  $2c$ , respectively, as labeled in the Figure 5.31. Through focus  $H$  we draw a parallel to  $DK$  (light dotted line) to determine point of intersection  $I$  with segment  $SP$ . Through point  $Q$  we draw a parallel to  $DK$  (light solid line) to determine point of intersection  $V$  with segment  $CP$  (see the inset in Figure 5.31).

# Newton's Lemma, continued 2

**Proof (continued).**

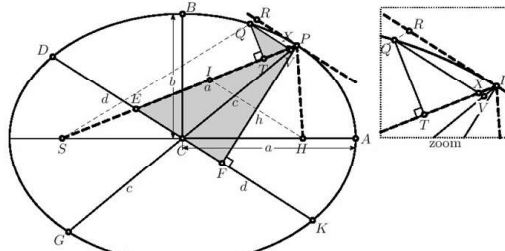


Fig. 5.31. Newton's proof of his lemma

By Apollonius III.48 (see [Section 3.2. The Ellipse](#)), the normal to  $DK$  from point  $F$  to point  $P$  (let  $PF = h$ ) is the bisector of angle  $SPH$ . That is, triangle  $IPH$  is an isosceles triangle and so (by Euclid I.6)  $IP = PH$ . Since  $SC = CH$ , then by Thales' Intercept Theorem (Theorem 1.1) we have  $SE = EI$ . By Apollonius III.52 (the alternative definition of an ellipse from [Section 3.2. The Ellipse](#)), we have  $SE + EI + IP + PH = 2a$ . Substituting  $EI$  for  $SE$  and substituting  $IP$  for  $PH$ , we have:  $EP = EI + IP = a$ . (5.55)

# Newton's Lemma, continued 3

**Proof (continued).**

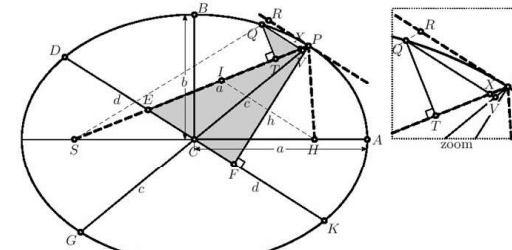
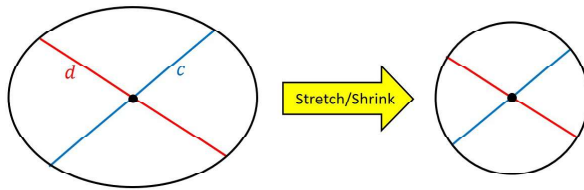


Fig. 5.31. Newton's proof of his lemma

If the ellipse were in fact a circle, then by Euclid III.35 (see [Section 2.2. Book III](#)) we would have  $GV \cdot VP = QV^2$  (here we would need to extend  $QV$  until it is a chord of the circle; see the inset in Figure 5.31).

## Newton's Lemma, continued 4

Proof (continued).



We now take the ellipse (along with the segments  $GV$ ,  $VP$ , and  $QV$ ) and stretch/shrink it in the direction of the blue "diameter" by a factor  $1/c$ , and stretch/contract it in the direction of the red "diameter" by a factor of  $1/d$ . This results in a circle of radius 1. Since  $GV$  and  $VP$  lie along the blue diameter then these are transformed to the circle with lengths  $GV/c$  and  $VP/c$ . Since  $QV$  lies on a line parallel to the blue diameter then this is transformed to the circle with length  $QV/d$ . So from the above equation  $GV \cdot VP = QV^2$  for the chords of a circle, we have

$$\frac{GV \cdot VP}{c^2} = \frac{QV^2}{d^2}, \text{ or: (3) } VP = \frac{c^2}{GV} \cdot \frac{QV^2}{d^2}.$$

## Newton's Lemma, continued 5

Proof (continued).

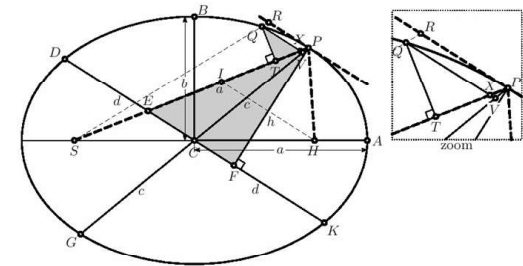


Fig. 5.31. Newton's proof of his lemma

Next, we express  $VP$  in terms of  $RQ$ , and express  $QV$  in terms of  $QT$  (see the inset). Now triangle  $XVP$  is similar to triangle  $ECP$ , because they share an angle at point  $P$  and angles  $PXV$  and  $PEC$  are corresponding angles for parallel line segments  $QV$  and  $EF$  with transversal  $CP$  (see Figure 1.7 center in [Section 1.3. Properties of Angles](#)); so the three angles of triangles  $XVP$  and  $ECP$  are the same. So  $XP/VP = EP/CP$  or (since  $EP = a$  by (5.55))  $XP/VP = a/c$ , or: (2)  $XP = VP \cdot a/c$ .

## Newton's Lemma, continued 6

Proof (continued).

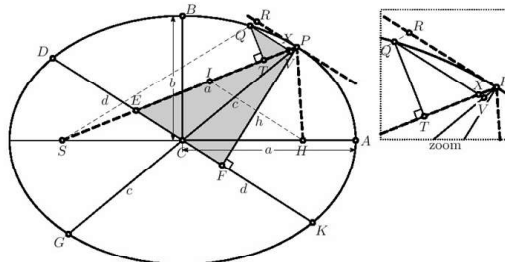


Fig. 5.31. Newton's proof of his lemma

Triangle  $QTX$  is similar to triangle  $PFE$  since both of these are right triangles and angles  $QXT$  and  $PEF$  are alternate interior angles (or "parallel angles," see Figure 1.7 let in [Section 1.3. Properties of Angles](#)) for parallel line segments  $QX$  and  $EF$  with transversal  $EX$ ; so the three angles of triangles  $QTX$  and  $PFE$  are the same. Also  $QX/QT = PE/PF$  or (since  $PF = h$ )  $QX/QT = a/h$ , or:

$$(6) \quad QX = QT \cdot a/h.$$

## Newton's Lemma, continued 7

Proof (continued).

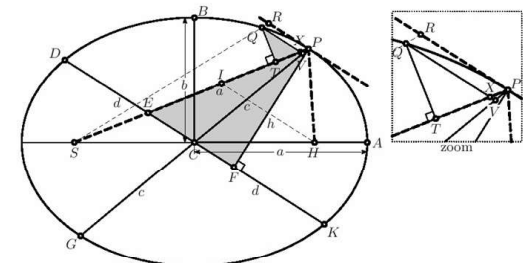


Fig. 5.31. Newton's proof of his lemma

We now consider  $PQ$  "infinitely small." In contemporary (rigorous) terms, we consider a limit as  $Q \rightarrow P$  and  $Q$  is on the ellipse. We see from the inset of Figure 5.31 that:

$$(1) \quad RQ \approx XP, \quad (4) \quad GV \approx GP = 2c, \quad (5) \quad QV \approx QX.$$

In fact, each of these can be written as equalities in the limit.

## Newton's Lemma, continued 8

**Proof (continued).** We now have:

$$\begin{aligned}
 RQ &\approx XP \text{ by (1)} \\
 &= VP \cdot \frac{a}{c} \text{ by (2)} \\
 &= \left( \frac{c^2}{GV} \cdot \frac{QV^2}{d^2} \right) \cdot \frac{a}{c} \text{ since by (3) } VP = \frac{c^2}{GV} \cdot \frac{QV^2}{d^2} \\
 &\approx \frac{c^2}{2c} \cdot \frac{QV^2}{d^2} \cdot \frac{a}{c} \text{ since by (4) } GV \approx GP = 2c \\
 &\approx \frac{c^2}{2c} \cdot \frac{QX^2}{d^2} \cdot \frac{a}{c} \text{ since by (5) } QV \approx QX \\
 &= \frac{c^2}{2c} \cdot \frac{(QT \cdot a/h)^2}{d^2} \cdot \frac{a}{c} \text{ since by (6) } QX = QT \cdot a/h \\
 &= \frac{a^3}{2h^2d^2} \cdot QT^2.
 \end{aligned}$$

## Newton's Lemma, continued 10

**Newton's Lemma.** Let  $APQ$  be an ellipse with focus  $S$  and suppose  $P$  to be the position of the planet moving towards  $Q$ , while the point  $R$  moves on the tangent with  $S, Q, P$  collinear. Let  $T$  be the orthogonal projection of  $Q$  onto  $PS$  (see Figure 5.30, right). Then if the distance  $PQ$  tends to zero, we have  $RQ \approx (\text{Constant}) \cdot QT^2$ , where the constant is independent of the position of  $P$  on the ellipse.

**Proof (continued).** So with  $hd = ab$ , we have from the approximation

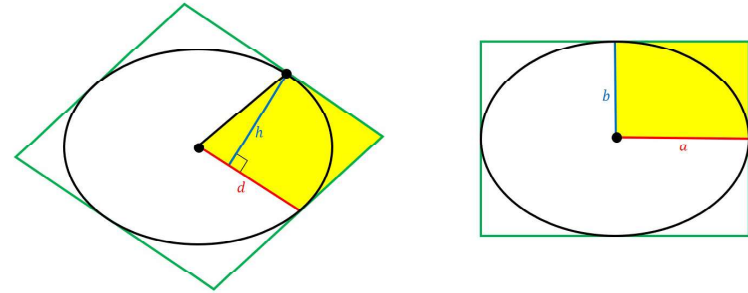
$$RQ \approx \frac{a^3}{2h^2d^2} \cdot QT^2 \text{ (established above) that}$$

$$RQ \approx \frac{a^3}{2h^2d^2} \cdot QT^2 = \frac{a^3}{2a^2b^2} \cdot QT^2 = \frac{a}{2b^2} \cdot QT^2.$$

That is,  $RQ \approx (\text{Constant}) \cdot QT^2$  where  $\text{Constant} = \frac{a}{2b^2}$  and the constant is independent of the position of  $P$  on the ellipse, as claimed.  $\square$

## Newton's Lemma, continued 9

**Proof (continued).** We now apply Apollonius VII.31, which is stated in Exercise 3.5.2 as: "All parallelograms circumscribed about any conjugate diameters of a given ellipse are equal."

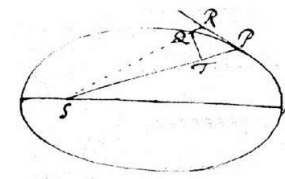


Notice that 1/4 of the areas of the inscribed rectangles above are  $hd$  (left) and  $ab$  (right). Since the areas of the parallelograms are the same by Apollonius VII.31, then  $hd = ab$ .

## Theorem 5.9. (Proposition 11 of Newton's *Principia Mathematica*)

**Theorem 5.9.** (Proposition 11 of Newton's *Principia Mathematica*.) A body  $P$ , orbiting according to Kepler 1 and 2 [i.e., Kepler's 1st and 2nd Laws], moves under the effect of a centripetal force, directed to the centre  $S$ , satisfying the law  $f = \frac{\text{Constant}}{r^2}$ , where  $r$  is the distance  $SP$ .

**Proof.** By equation (5.53) we have that the force  $f$  is proportional to  $RQ$ . By Newton's Lemma,  $RQ$  is approximately proportional to  $QT^2$  (and so  $f$  is approximately proportional to  $QT^2$ ). The area of the triangle  $SPQ$  in Figure 5.30 right is  $SP \cdot QT/2$  and this area is a constant for a fixed  $\Delta t$  by Kepler's Second Law.



## Theorem 5.9. (Proposition 11 of Newton's *Principia Mathematica*, continued)

**Theorem 5.9.** (Proposition 11 of Newton's *Principia Mathematica*.)

A body  $P$ , orbiting according to Kepler 1 and 2 [i.e., Kepler's 1st and 2nd Laws], moves under the effect of a centripetal force, directed to the centre  $S$ , satisfying the law  $f = \frac{\text{Constant}}{r^2}$ , where  $r$  is the distance  $SP$ .

**Proof (continued).** Since  $SP \cdot QT/2$  is constant, then  $QT$  is inversely proportional to  $SP$ . Hence  $f$  is approximately inversely proportional to  $SP = r$ . That is,  $f \approx \frac{\text{Constant}}{r^2}$ . By taking a limit as  $\Delta t \rightarrow 0$ , we have  $Q \rightarrow P$  and the approximation becomes precise.  $\square$