History of Geometry

Chapter 5. Trigonometry

5.10. The Great Discoveries of Kepler and Newton—Proofs of Theorems

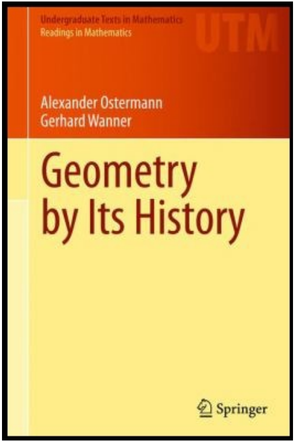


Table of contents

Theorem 5.8 (Newton's Theorem 1)

Theorem 5.8. (Theorem 1 of Newton's *Principia Mathematica*.) "The areas, which revolving bodies describe by radii drawn to an immoveable centre of force, do lie in some immoveable planes, and are proportional to the times in which they are described."

Proof. We go through the argument given in Ostermann and Wanner (which is the same as given by Newton in *Principia*). The argument is a discretized version of the physical problem. Newton appeals to another result (Lemma III in his Book I) to move from the discrete to the continuous case.

Theorem 5.8 (Newton's Theorem 1)

Theorem 5.8. (Theorem 1 of Newton's *Principia Mathematica*.) "The areas, which revolving bodies describe by radii drawn to an immoveable centre of force, do lie in some immoveable planes, and are proportional to the times in which they are described."

Proof. We go through the argument given in Ostermann and Wanner (which is the same as given by Newton in *Principia*). The argument is a discretized version of the physical problem. Newton appeals to another result (Lemma III in his Book I) to move from the discrete to the continuous case. We place the Sun at point S (the "centre of force") and approximate the orbit with points A, B, C, D, E, etc. The force f then acts on the object orbiting the Sun by pulling the object in the direction of the Sun. We let the force act on the object at point A and then assume the object moves in a straight line over time period Δt without the form acting from A to B (the motion is along a straight line by Newton's First Law, or "Lex 1"). See the figure below.

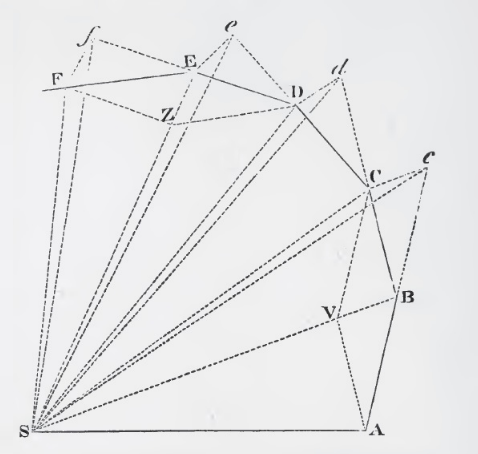
Theorem 5.8 (Newton's Theorem 1)

Theorem 5.8. (Theorem 1 of Newton's *Principia Mathematica*.) "The areas, which revolving bodies describe by radii drawn to an immoveable centre of force, do lie in some immoveable planes, and are proportional to the times in which they are described."

Proof. We go through the argument given in Ostermann and Wanner (which is the same as given by Newton in *Principia*). The argument is a discretized version of the physical problem. Newton appeals to another result (Lemma III in his Book I) to move from the discrete to the continuous case. We place the Sun at point S (the "centre of force") and approximate the orbit with points A, B, C, D, E, etc. The force f then acts on the object orbiting the Sun by pulling the object in the direction of the Sun. We let the force act on the object at point A and then assume the object moves in a straight line over time period Δt without the form acting from A to B (the motion is along a straight line by Newton's First Law, or "Lex 1"). See the figure below.

Theorem 5.8 (Newton's Theorem 1, continued 1)

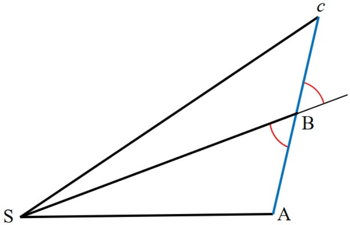
Proof (continued). Next, we let the force act again at point B by an amount $f\,\Delta t$. Now consider the triangles ABS and BcS in the figure. We claim that these triangles have the same areas. We argue this below.

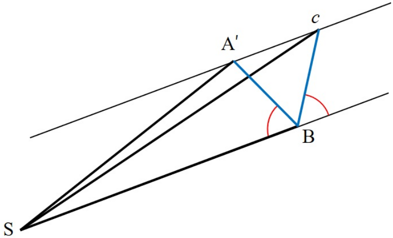


The Figure for Newton's Theorem 1 from Motte's 1846 translation of *Principia*.

Theorem 5.8 (Newton's Theorem 1, continued 2)

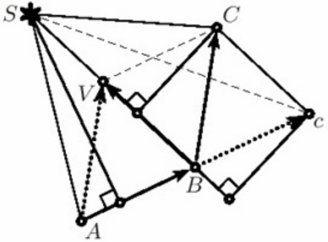
Proof (continued). To see that triangles *ABS* and *BcS* have the same area, reflect triangle *ABS* about the line containing points *S* and *B* and let A' denote the image of *A*. We then see that the triangle have a common base *SB* and the same altitude. So the areas are the same (by Euclid I.41).





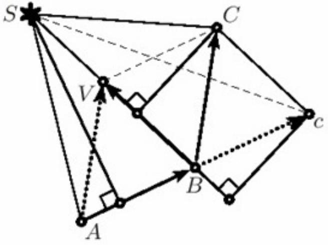
Theorem 5.8 (Newton's Theorem 1, continued 3)

Proof (continued). Next, we let the force affect the object at point B. By Newton's Second Law (or Lex 2), the force acts along the line from B to S; the "change of motion" (as it's called in Lex 2) is represented as the vector from B to V (we denote this vector, and similar vectors, as $\overrightarrow{BV}$) in Figure 5.29 (right) below. Adding this change in motion to the velocity of the object before the application of the force (represented by the vector $\overrightarrow{AB}$ in the figure; think of it as the change in position over time Δt), we get the new velocity by adding vectors $\overrightarrow{AB}$ and $\overrightarrow{BV}$ to get the resultant vector $\overrightarrow{AV}$. We translate $\overrightarrow{AV}$ to point B to get velocity vector $\overrightarrow{BC}$.



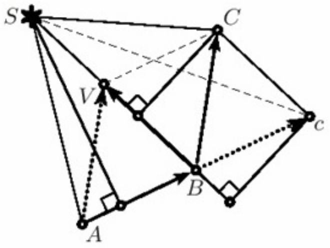
Theorem 5.8 (Newton's Theorem 1, continued 3)

Proof (continued). Next, we let the force affect the object at point B. By Newton's Second Law (or Lex 2), the force acts along the line from B to S; the "change of motion" (as it's called in Lex 2) is represented as the vector from B to V (we denote this vector, and similar vectors, as $\overrightarrow{BV}$) in Figure 5.29 (right) below. Adding this change in motion to the velocity of the object before the application of the force (represented by the vector $\overrightarrow{AB}$ in the figure; think of it as the change in position over time Δt), we get the new velocity by adding vectors $\overrightarrow{AB}$ and $\overrightarrow{BV}$ to get the resultant vector $\overrightarrow{AV}$. We translate $\overrightarrow{AV}$ to point B to get velocity vector $\overrightarrow{BC}$.



Theorem 5.8 (Newton's Theorem 1, continued 4)

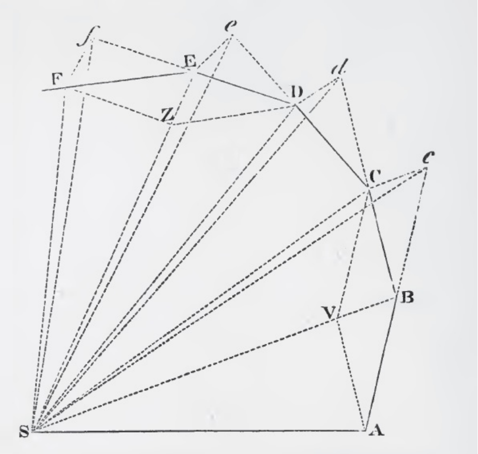
Proof (continued). By construction above, vector $\overrightarrow{AB}$ equals vector $\overrightarrow{BC}$. So when translating vectors $\overrightarrow{AB}$ and $\overrightarrow{AV}$ to point B, we see that segment Cc is parallel to segment SB. So the area of triangles BCS and BcS are the same since they have the same base, SB, and the same heights (represented, respectively, in the figure by the segments with the right-angle symbols which end at C and c; we are again using Euclid I.41 here).



Theorem 5.8 (Newton's Theorem 1, continued 5)

Proof (continued). Since, as shown above, the triangles ABS and BcS have the same areas, then we have that triangles ABS and BCS have the same areas. Similar, the areas of triangles ABS, BCS, CDS, DES, etc. are all the same.

So the claim holds when we have the discrete version of the problem with the force applied as impulses at equal time steps Δt.

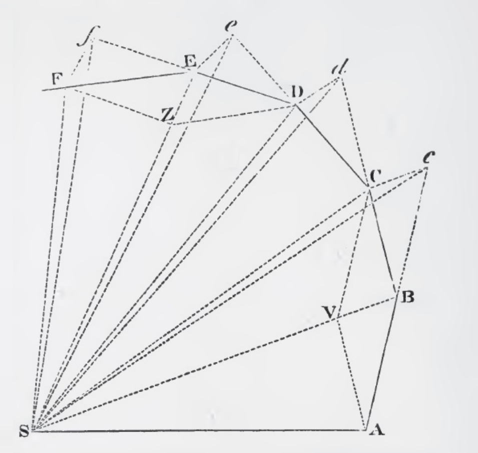


The Figure for Newton's Theorem 1 from Motte's 1846 translation of *Principia*.

Theorem 5.8 (Newton's Theorem 1, continued 5)

Proof (continued). Since, as shown above, the triangles ABS and BcS have the same areas, then we have that triangles ABS and BCS have the same areas. Similar, the areas of triangles ABS, BCS, CDS, DES, etc. are all the same.

So the claim holds when we have the discrete version of the problem with the force applied as impulses at equal time steps Δt.



The Figure for Newton's Theorem 1 from Motte's 1846 translation of *Principia*.

Theorem 5.8 (Newton's Theorem 1, continued 6)

Theorem 5.8. (Theorem 1 of Newton's *Principia Mathematica*.) "The areas, which revolving bodies describe by radii drawn to an immoveable centre of force, do lie in some immoveable planes, and are proportional to the times in which they are described."

Proof (continued). To complete the proof, Newton now applies his Lemma III which, in essence, involve taking a limit as $\Delta t \to 0$. Ostermann and Wanner describe Newton's approach as similar to the numerical technique called the "Euler method" which is used to approximate solutions to differential equations.

In the spirit of Calculus 1 (MATH 1910), we could consider the area swept out by the line segment joining the Sun and the orbiting object as a function time $A(t)$. We then have from Newton's discrete argument that $\Delta A/\Delta t$ is a constant. So we have $\displaystyle\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt}$ is constant (this is the constant of proportionality in Newton's theorem). □

Theorem 5.8 (Newton's Theorem 1, continued 6)

Theorem 5.8. (Theorem 1 of Newton's *Principia Mathematica*.) "The areas, which revolving bodies describe by radii drawn to an immoveable centre of force, do lie in some immoveable planes, and are proportional to the times in which they are described."

Proof (continued). To complete the proof, Newton now applies his Lemma III which, in essence, involve taking a limit as $\Delta t \to 0$. Ostermann and Wanner describe Newton's approach as similar to the numerical technique called the "Euler method" which is used to approximate solutions to differential equations.

In the spirit of Calculus 1 (MATH 1910), we could consider the area swept out by the line segment joining the Sun and the orbiting object as a function time $A(t)$. We then have from Newton's discrete argument that $\Delta A/\Delta t$ is a constant. So we have $\displaystyle\lim_{\Delta t \to 0} \frac{\Delta A}{\Delta t} = \frac{dA}{dt}$ is constant (this is the constant of proportionality in Newton's theorem). □

Newton's Lemma

Newton's Lemma. Let APQ be an ellipse with focus S and suppose P to be the position of the planet moving towards Q, while the point R moves on the tangent with S, Q, P collinear. Let T be the orthogonal projection of Q onto PS (see Figure 5.30, right). Then if the distance PQ tends to zero, we have $RQ \approx (\text{Constant}) \cdot QT^2$, where the constant is independent of the position of P on the ellipse.

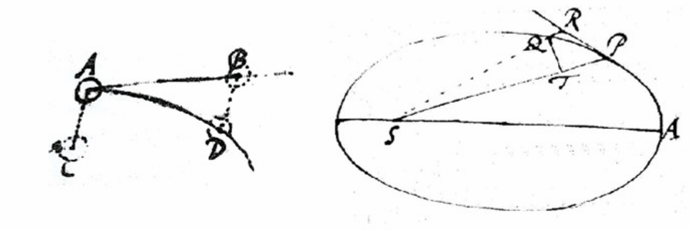


Fig. 5.30. Reproductions from Newton's autograph (1684), manuscript Cambridge Univ. Lib. Add. 3965[6]; the force acting on a moving body (left); picture for Newton's lemma (right). Reproduced by kind permission of the Syndics of Cambridge University Library

Newton's Lemma, continued 1

Proof.

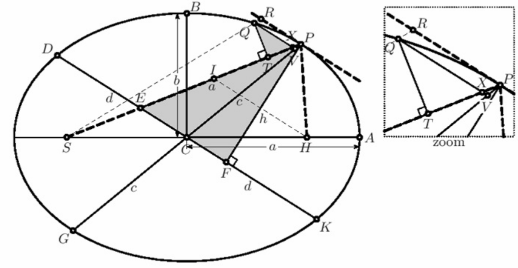


Fig. 5.31. Newton's proof of his lemma

From this figure, we have by Apollonius II.6 (see Section 3.2. The Ellipse, Figure 3.7(b)) that the tangent PR is parallel to the "diameter" DCK which is conjugate to diameter GCP. Let the lengths of these diameters be $2d$ and $2c$, respectively, as labeled in the Figure 5.31. Through focus H we draw a parallel to DK (light dotted line) to determine point of intersection I with segment SP. Through point Q we draw a parallel to DK (light solid line) to determine point of intersection V with segment CP (see the inset in Figure 5.31).

Newton's Lemma, continued 1

Proof.

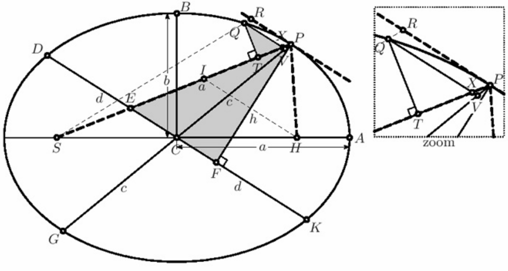


Fig. 5.31. Newton's proof of his lemma

From this figure, we have by Apollonius II.6 (see Section 3.2. The Ellipse, Figure 3.7(b)) that the tangent PR is parallel to the "diameter" DCK which is conjugate to diameter GCP. Let the lengths of these diameters be $2d$ and $2c$, respectively, as labeled in the Figure 5.31. Through focus H we draw a parallel to DK (light dotted line) to determine point of intersection I with segment SP. Through point Q we draw a parallel to DK (light solid line) to determine point of intersection V with segment CP (see the inset in Figure 5.31).

Newton's Lemma, continued 2

Proof (continued).

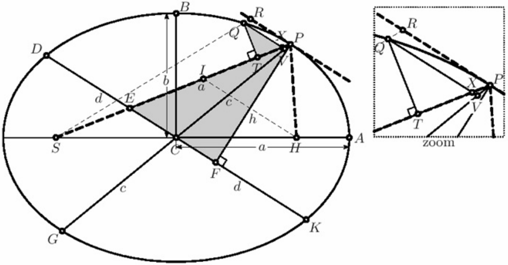


Fig. 5.31. Newton's proof of his lemma

By Apollonius III.48 (see Section 3.2. The Ellipse), the normal to DK from point F to point P (let $PF = h$) is he bisector of angle SPH. That is, triangle IPH is an isosceles triangle and so (by Euclid I.6) $IP = PH$. Since $SC = CH$, then by Thales' Intercept Theorem (Theorem 1.1) we have $SE = EI$. By Apollonius III.52 (the alternative definition of an ellipse from Section 3.2. The Ellipse), we have $SE + EI + IP + PH = 2a$. Substituting EI for SP and substituting IP for PH, we have: $EP = EI + IP = a$. (5.55)

Newton's Lemma, continued 2

Proof (continued).

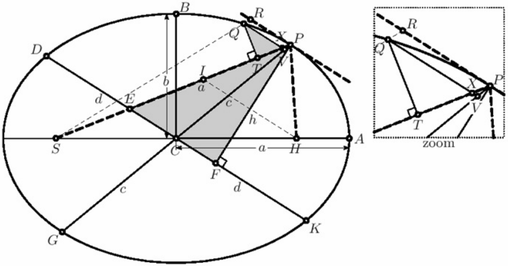


Fig. 5.31. Newton's proof of his lemma

By Apollonius III.48 (see Section 3.2. The Ellipse), the normal to DK from point F to point P (let $PF = h$) is he bisector of angle SPH. That is, triangle IPH is an isosceles triangle and so (by Euclid I.6) $IP = PH$. Since $SC = CH$, then by Thales' Intercept Theorem (Theorem 1.1) we have $SE = EI$. By Apollonius III.52 (the alternative definition of an ellipse from Section 3.2. The Ellipse), we have $SE + EI + IP + PH = 2a$. Substituting EI for SP and substituting IP for PH, we have: $EP = EI + IP = a$. (5.55)

Newton's Lemma, continued 3

Proof (continued).

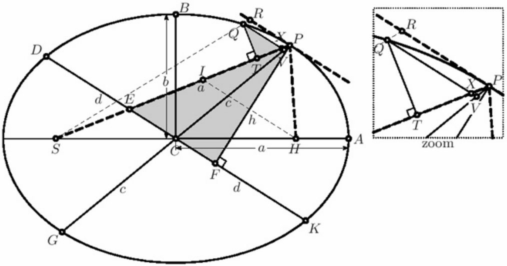
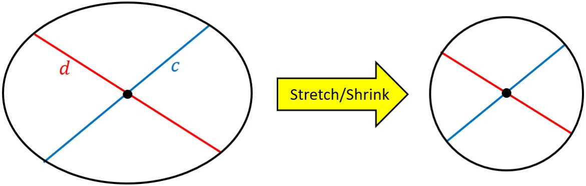


Fig. 5.31. Newton's proof of his lemma

If the ellipse were in fact a circle, then by Euclid III.35 (see Section 2.2. Book III) we would have $GV \cdot VP = QV^2$ (here we would need to extend QV until it is a chord of the circle; see the inset in Figure 5.31).

Newton's Lemma, continued 4

Proof (continued).

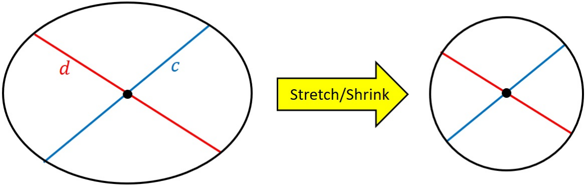


We now take the ellipse (along with the segments GV, VP, and QV) and stretch/shrink it in the direction of the blue "diameter" by a factor $1/c$, and stretch/contract it in the direction of the red "diameter" by a factor of $1/d$. This results in a circle of radius 1. Since GV and VP lie along the blue diameter then these are transformed to the circle with lengths GV/c and VP/c. Since QV lies on a line parallel to the blue diameter then this is transformed to the circle with length QV/d. So from the above equation $GV \cdot VP = QV^2$ for the chords of a circle, we have

$$\frac{GV \cdot VP}{c^2} = \frac{QV^2}{d^2}, \text{ or: (3) } VP = \frac{c^2}{GV} \cdot \frac{QV^2}{d^2}.$$

Newton's Lemma, continued 4

Proof (continued).



We now take the ellipse (along with the segments GV, VP, and QV) and stretch/shrink it in the direction of the blue "diameter" by a factor $1/c$, and stretch/contract it in the direction of the red "diameter" by a factor of $1/d$. This results in a circle of radius 1. Since GV and VP lie along the blue diameter then these are transformed to the circle with lengths GV/c and VP/c. Since QV lies on a line parallel to the blue diameter then this is transformed to the circle with length QV/d. So from the above equation $GV \cdot VP = QV^2$ for the chords of a circle, we have

$$\frac{GV \cdot VP}{c^2} = \frac{QV^2}{d^2}, \text{ or: (3) } VP = \frac{c^2}{GV} \cdot \frac{QV^2}{d^2}.$$

Newton's Lemma, continued 5

Proof (continued).

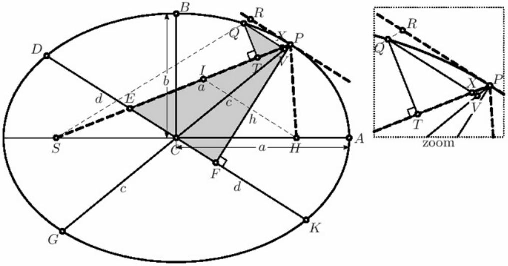


Fig. 5.31. Newton's proof of his lemma

Next, we express VP in terms of RQ, and express QV in terms of QT (see the inset). Now triangle XVP is similar to triangle ECP, because they share an angle at point P and angles PXV and PEC are corresponding angles for parallel line segments QV and EF with transversal CP (see Figure 1.7 center in Section 1.3. Properties of Angles); so the three angles of triangles XVP and ECP are the same. So $XP/VP = EP/CP$ or (since $EP = a$ by (5.55)) $XP/VP = a/c$, or: (2) $XP = VP \cdot a/c$.

Newton's Lemma, continued 6

Proof (continued).

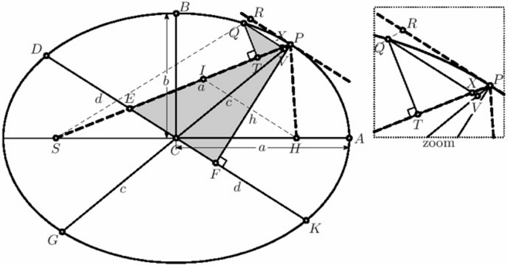


Fig. 5.31. Newton's proof of his lemma

Triangle QTX is similar to triangle PFE since both of these are right triangles and angles QXT and PEF are alternate interior angles (or "parallel angles," see Figure 1.7 let in Section 1.3. Properties of Angles) for parallel line segments QX and EF with transversal EX; so the three angles of triangles QTX and PFE are the same. Also $QX/QT = PE/PF$ or (since $PF = h$) $QX/QT = a/h$, or:

$$(6)\ QX = QT \cdot a/h.$$

Newton's Lemma, continued 7

Proof (continued).

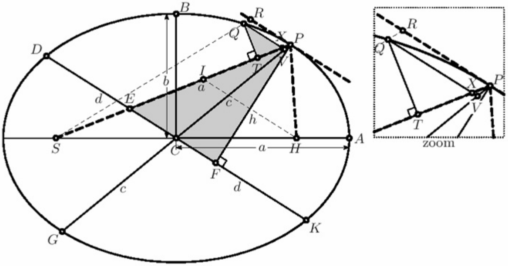


Fig. 5.31. Newton's proof of his lemma

We now consider PQ "infinitely small." In contemporary (rigorous) terms, we consider a limit as $Q \to P$ and Q is on the ellipse. We see from the inset of Figure 5.31 that:

$$(1)\, RQ \approx XP, \;\; (4)\, GV \approx GP = 2c, \;\; (5)\, QV \approx QX.$$

In fact, each of these can be written as equalities in the limit.

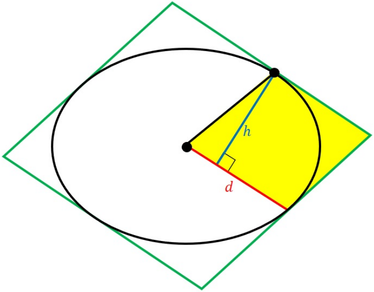
Newton's Lemma, continued 8

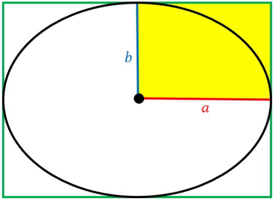
Proof (continued). We now have:

$$\begin{aligned}
RQ &\approx XP \text{ by } (1) \\
&= VP \cdot \frac{a}{c} \text{ by } (2) \\
&= \left(\frac{c^2}{GV} \cdot \frac{QV^2}{d^2}\right) \cdot \frac{a}{c} \text{ since by } (3)\ VP = \frac{c^2}{GV} \cdot \frac{QV^2}{d^2} \\
&\approx \frac{c^2}{2c} \cdot \frac{QV^2}{d^2} \cdot \frac{a}{c} \text{ since by } (4)\ GV \approx GP = 2c \\
&\approx \frac{c^2}{2c} \cdot \frac{QX^2}{d^2} \cdot \frac{a}{c} \text{ since by } (5)\ QV \approx QX \\
&= \frac{c^2}{2c} \cdot \frac{(QT \cdot a/h)^2}{d^2} \cdot \frac{a}{c} \text{ since by } (6)\ QX = QT \cdot a/h \\
&= \frac{a^3}{2h^2 d^2} \cdot QT^2.
\end{aligned}$$

Newton's Lemma, continued 9

Proof (continued). We now apply Apollonius VII.31, which is stated in Exercise 3.5.2 as: "All parallelograms circumscribed about any conjugate diameters of a given ellipse are equal."





Notice that 1/4 of the areas of the inscribed rectangles above are hd (left) and ab (right). Since the areas of the parallelograms are the same by Apollonius VII.31, then $hd = ab$.

Newton's Lemma, continued 10

Newton's Lemma. Let APQ be an ellipse with focus S and suppose P to be the position of the planet moving towards Q, while the point R moves on the tangent with S, Q, P collinear. Let T be the orthogonal projection of Q onto PS (see Figure 5.30, right). Then if the distance PQ tends to zero, we have $RQ \approx (\text{Constant}) \cdot QT^2$, where the constant is independent of the position of P on the ellipse.

Proof (continued). So with $hd = ab$, we have from the approximation $RQ \approx \dfrac{a^3}{2h^2d^2} \cdot QT^2$ (established above) that

$$RQ \approx \frac{a^3}{2h^2d^2} \cdot QT^2 = \frac{a^3}{2a^2b^2} \cdot QT^2 = \frac{a}{2b^2} \cdot QT^2.$$

That is, $RQ \approx (\text{Constant}) \cdot QT^2$ where Constant $= \dfrac{a}{2b^2}$ and the constant is independent of the position of P on the ellipse, as claimed. □

Theorem 5.9. (Proposition 11 of Newton's *Principia Mathematica*)

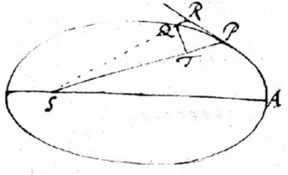
Theorem 5.9. (Proposition 11 of Newton's *Principia Mathematica*.) A body P, orbiting according to Kepler 1 and 2 [i.e., Kepler's 1st and 2nd Laws], moves under the effect of a centripetal force, directed to the centre S, satisfying the law $f = \dfrac{Constant}{r^2}$, where r is the distance SP.

Proof. By equation (5.53) we have that the force f is proportional to RQ. By Newton's Lemma, RQ is approximately proportional to QT^2 (and so f is approximately proportional to QT^2). The area of the triangle SPQ in Figure 5.30 right is $SP \cdot QT/2$ and this area is a constant for a fixed Δt, as seen in the proof of Kepler's Second Law (Theorem 5.8).

Theorem 5.9. (Proposition 11 of Newton's *Principia Mathematica*)

Theorem 5.9. (Proposition 11 of Newton's *Principia Mathematica*.) A body P, orbiting according to Kepler 1 and 2 [i.e., Kepler's 1st and 2nd Laws], moves under the effect of a centripetal force, directed to the centre S, satisfying the law $f = \dfrac{Constant}{r^2}$, where r is the distance SP.

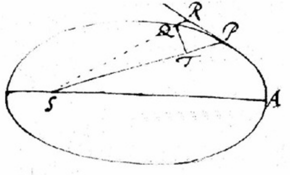
Proof. By equation (5.53) we have that the force f is proportional to RQ. By Newton's Lemma, RQ is approximately proportional to QT^2 (and so f is approximately proportional to QT^2). The area of the triangle SPQ in Figure 5.30 right is $SP \cdot QT/2$ and this area is a constant for a fixed Δt, as seen in the proof of Kepler's Second Law (Theorem 5.8).



Theorem 5.9. (Proposition 11 of Newton's *Principia Mathematica*)

Theorem 5.9. (Proposition 11 of Newton's *Principia Mathematica*.) A body P, orbiting according to Kepler 1 and 2 [i.e., Kepler's 1st and 2nd Laws], moves under the effect of a centripetal force, directed to the centre S, satisfying the law $f = \dfrac{Constant}{r^2}$, where r is the distance SP.

Proof. By equation (5.53) we have that the force f is proportional to RQ. By Newton's Lemma, RQ is approximately proportional to QT^2 (and so f is approximately proportional to QT^2). The area of the triangle SPQ in Figure 5.30 right is $SP \cdot QT/2$ and this area is a constant for a fixed Δt, as seen in the proof of Kepler's Second Law (Theorem 5.8).



Theorem 5.9. (Proposition 11 of Newton's *Principia*, continued)

Theorem 5.9. (Proposition 11 of Newton's *Principia Mathematica*.) A body P, orbiting according to Kepler 1 and 2 [i.e., Kepler's 1st and 2nd Laws], moves under the effect of a centripetal force, directed to the centre S, satisfying the law $f = \frac{Constant}{r^2}$, where r is the distance SP.

Proof (continued). Since $SP \cdot QT/2$ is constant, then QT is inversely proportional to SP. Hence f is approximately inversely proportional to $SP = r$. That is, $f \approx \frac{Constant}{r^2}$. By taking a limit as $\Delta t \to 0$, we have $Q \to P$ and the approximation becomes precise. □