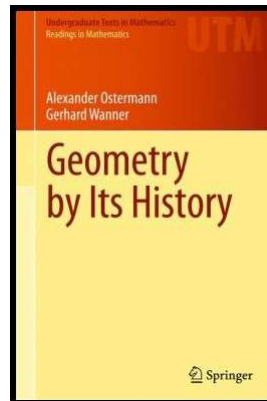


History of Geometry

Chapter 5. Trigonometry

5.2. Regiomontanus and Euler's Trigonometric Functions—Proofs of Theorems

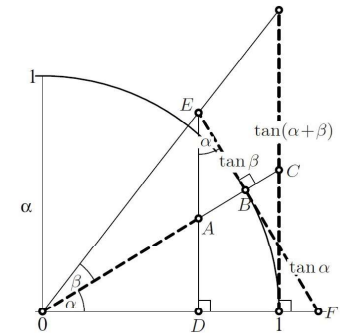


Theorem 5.2. Addition Formulas

Theorem 5.2. (Addition Formulas). The following identities hold:

$$\begin{aligned} \sin(\alpha + \beta) &= \sin \alpha \cos \beta + \cos \alpha \sin \beta \\ \cos(\alpha + \beta) &= \cos \alpha \cos \beta - \sin \alpha \sin \beta \\ \tan(\alpha + \beta) &= \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}. \end{aligned}$$

Proof. Geometric proofs of the summation formulas for sine and cosine are to be given in Exercise 5.3. The summation formula for tangent results by dividing the one for sine by the one for cosine. However, we now give a direct geometric proof independent of the proofs for sine and cosine. Consider the figure at the right. We give a geometric proof for α an acute angle, and the general case will follow.



Theorem 5.2. Addition Formulas, continued

Proof. Since angles $\angle OAD$ and $\angle EAB$ are opposite angle, then they are congruent. Since $\triangle ADO$ and $\triangle ABE$ are right triangles with two equal acute angles, then they are similar and so $\angle AEB = \alpha$.

In $\triangle ABE$ we have $\tan \alpha = AB/OA$ or $AB = \tan \alpha \cdot OA$, so that $OA = 1 - AB = 1 - \tan \alpha \tan \beta$. By Thales Intercept Theorem (Theorem 1.1) $\tan(\alpha + \beta) = (ED/OA)(OC)$ and $(OC)/(1) = (EF)/(ED)$.

Therefore $\tan(\alpha + \beta) = (EF)/(OA)$. We see from $\triangle ABE$ that $EB = \tan \beta$, and from $\triangle OBF$ that $BF = \tan \alpha$. Hence $EF = EB + BF = \tan \alpha + \tan \beta$.

We now have $\tan(\alpha + \beta) = \frac{EF}{OA} = \frac{\tan \alpha + \tan \beta}{1 - \tan \alpha \tan \beta}$, as claimed. \square

