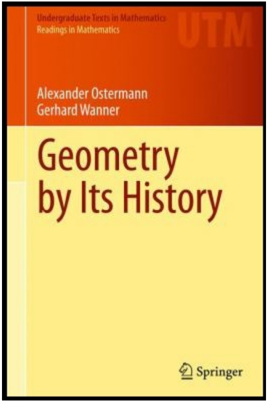
History of Geometry

Chapter 5. Trigonometry

5.2. Regiomontanus and Euler's Trigonometric Functions—Proofs of Theorems



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Proof. Geometric proofs of the summation formulas for sine and cosine are to be given in Exercise 5.3. The summation formula for tangent results by dividing the dividing the one for sine by the one for cosine. However, we now give a direct geometric proof independent of the proofs for sine and cosine.

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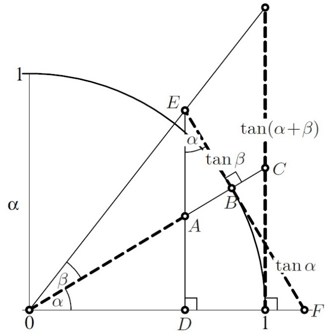

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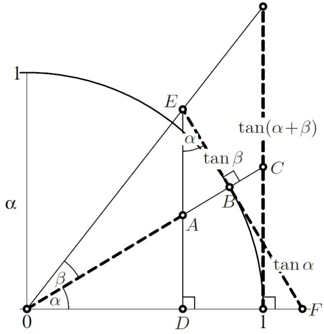
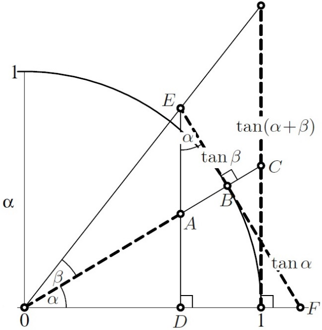


Theorem 5.2. Addition Formulas, continued

Proof. Since angles $\angle 0AD$ and $\angle EAB$ are opposite angle, then they are congruent. Since $\triangle AD0$ and $\triangle ABE$ are right triangles triangles with two equal acute angles, then they are similar and so $\measuredangle AEB = \alpha$. In $\triangle ABE$ we have $\tan\alpha = AB/\tan\beta$ or $AB = \tan\alpha\tan\beta$, so that $0A = 1 - AB = 1 - \tan\alpha\tan\beta$. By Thales Intercept Theorem (Theorem 1.1) $\tan(\alpha+\beta) = (ED/0A)(0C)$ and $(0C)/(1) = (EF)/(ED)$.



Therefore $\tan(\alpha+\beta) = (EF)/(0A)$. We see from $\triangle ABE$ that $EB = \tan\beta$, and from $\triangle 0BF$ that $BF = \tan\alpha$. Hence $EF = EB + BF = \tan\alpha + \tan\beta$. We now have $\tan(\alpha+\beta) = \dfrac{EF}{0A} = \dfrac{\tan\alpha + \tan\beta}{1 - \tan\alpha + \tan\beta}$, as claimed. □

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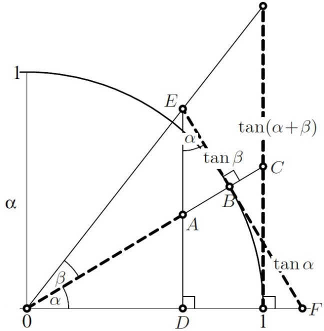

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