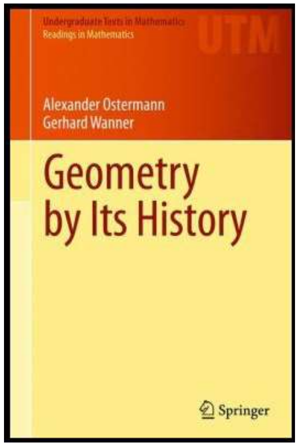
# History of Geometry

**Chapter 5. Trigonometry**

5.9. Trigonometric Formulation for Conics—Proofs of Theorems








## Theorem 5.9.A

**Theorem 5.9.A.** With the parameters introduced above and in Figure 5.26 we have the relations $r = \dfrac{p}{1 + e\cos\varphi}$ and $r = a - ex = a - ae\cos u$ where $p$ is the vertical distance from a focus to the ellipse and $x$ is the directed distance of $P$ from the minor axis of the ellipse (when the ellipse has its major axis horizontal; see Figure 3.4 in Section 3.2. The Ellipse).

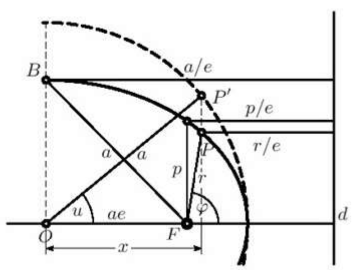
**Proof.** Recall that the sum of the distance of $P$ to the two foci is twice the lengthof the major axis (this is equation (3.5) in Section 3.2. The Ellipse). So the distance $BF$ is equal to $a$. The lengths $r/e$, $p/e$, and $a/e$ are given in Figure 3.4 (based on the definition of an ellipse in terms of a directrix and eccentricity), and also given in Figure 5.26.





## Theorem 5.9.A (continued 1)

**Proof (continued).**



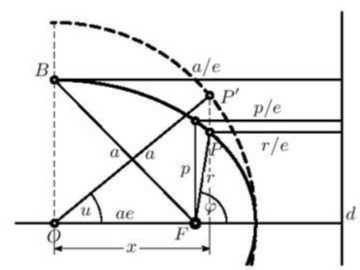

From Figure 5.26 we see that distance $p/e$ equals distance $r/e$ plus the base of the right triangle with hypotenuse $\overline{FP}$. The base has length $r\cos\varphi$ and so $\dfrac{p}{e} - r\cos\varphi + \dfrac{r}{e}$. That is, $p = er\cos\varphi + r = r(e\cos\varphi + 1)$ or $r = \dfrac{p}{1 + e\cos\varphi}$, as claimed.





## Theorem 5.9.A (continued 2)

**Proof (continued).**




Also from Figure 5.26, distance $a/e$ equals distance $r/e$ plus the base of the triangle with hypotenuse $OP'$. The base has length $a\cos u$ and so $\dfrac{a}{e} = \dfrac{r}{e} + a\cos u$. That is $a = r + ea\cos u$, or $r = a - ea\cos u$, as claimed. Also, $x = a\cos u$ so we also have $r = a - ex$, as claimed. □

## Theorem 5.9.B

**Theorem 5.9.B.** The area $\mathcal{A}$ swept out by the line joining the focus $F$ to a point $P$ on the ellipse over an angle $\varphi$ measured from the semimajor axis (see Figure 5.27, left) is

$$\mathcal{A} = \frac{ab}{2}(u - e\sin u).$$

**Proof.** We seek the shaded are $\mathcal{A}$ in Figure 5.27 (left).

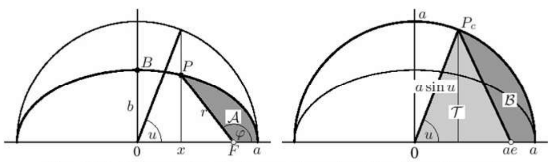



**Fig. 5.27.** Computation of the area $\mathcal{A}$ swept out by the radius vector

We stretch the ellipse vertically by a factor of $a/b$ (that is, the $y$-value of each point is multiplied by $a/b$).





## Theorem 5.9.B (continued 1)

**Proof (continued).**

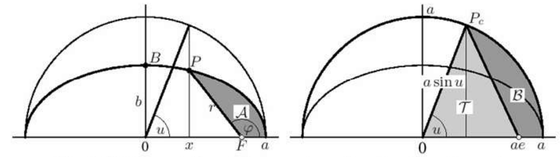



**Fig. 5.27.** Computation of the area $\mathcal{A}$ swept out by the radius vector

With $\mathcal{B}$ as the shaded area in Figure 5.27 (right) we then have $\mathcal{B} = \frac{a}{b}\mathcal{A}$ (the idea is similar to that of Theorem 1.6, though that does not rigorously justify this claim). The area of the sector in the circle with central angle $u$ measured in radians is $a^2u/2$. With $\mathcal{T}$ as the area of the triangle in Figure 5.27 (right), we have that $a^2u/2$ is then $\mathcal{T} + \mathcal{B}$. Since $\mathcal{T} = \frac{1}{2}(ae)(a\sin u) = \frac{1}{2}a^2 e\sin u$. Therefore,

$$\mathcal{B} = \frac{a^2u}{2} = \mathcal{T} = \frac{a^2u}{2} - \frac{a^2e\sin u}{2} = \frac{a^2}{2}(u - e\sin u), \ldots$$





## Theorem 5.9.B (continued 2)

**Theorem 5.9.B.** The area $\mathcal{A}$ swept out by the line joining the focus $F$ to a point $P$ on the ellipse over an angle $\varphi$ measured from the semimajor axis (see Figure 5.27, left) is

$$\mathcal{A} = \frac{ab}{2}(u - e\sin u).$$

**Proof (continued).** ...

$$\mathcal{B} = \frac{a^2u}{2} = \mathcal{T} = \frac{a^2u}{2} - \frac{a^2e\sin u}{2} = \frac{a^2}{2}(u - e\sin u),$$

and hence

$$\mathcal{A} = \left(\frac{b}{a}\right)\frac{a^2}{2}(u - e\sin u) = \frac{ab}{2}(u - e\sin u),$$

as claimed. □