History of Geometry

Chapter 5. Trigonometry

5.9. Trigonometric Formulation for Conics—Proofs of Theorems

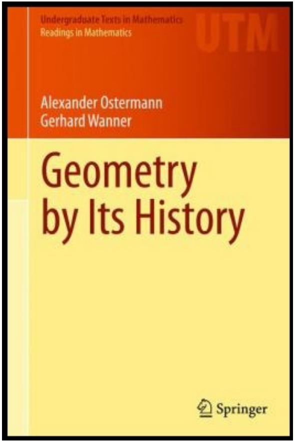

Table of contents

Theorem 5.9.A

Theorem 5.9.A. With the parameters introduced above and in Figure 5.26 we have the relations $r = \dfrac{p}{1 + e\cos\varphi}$ and $r = a - ex = a - ae\cos u$ where p is the vertical distance from a focus to the ellipse and x is the directed distance of P from the minor axis of the ellipse (when the ellipse has its major axis horizontal; see Figure 3.4 in Section 3.2. The Ellipse).

Proof. Recall that the sum of the distance of P to the two foci is twice the lengthof the major axis (this is equation (3.5) in Section 3.2. The Ellipse). So the distance BF is equal to a. The lengths r/e, p/e, and a/e are given in Figure 3.4 (based on the definition of an ellipse in terms of a directrix and eccentricity), and also given in Figure 5.26.

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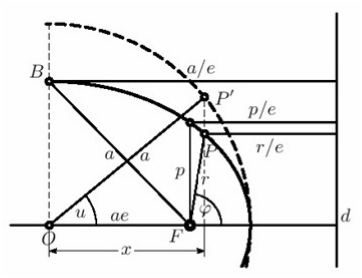
Theorem 5.9.A (continued 1)

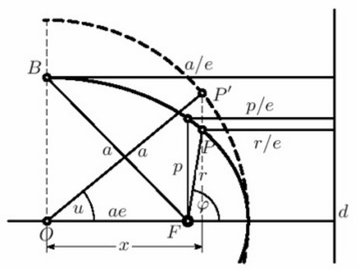
Proof (continued).



From Figure 5.26 we see that distance p/e equals distance r/e plus the base of the right triangle with hypotenuse $\overline{FP}$. The base has length $r\cos\varphi$ and so $\frac{p}{e} - r\cos\varphi + \frac{r}{e}$. That is, $p = er\cos\varphi + r = r(e\cos\varphi + 1)$ or $r = \dfrac{p}{1 + e\cos\varphi}$, as claimed.

Theorem 5.9.A (continued 2)

Proof (continued).



Also from Figure 5.26, distance a/e equals distance r/e plus the base of the triangle with hypotenuse OP'. The base has length $a\cos u$ and so $\dfrac{a}{e} = \dfrac{r}{e} + a\cos u$. That is $a = r + ea\cos u$, or $r = a - ea\cos u$, as claimed. Also, $x = a\cos u$ so we also have $r = a - ex$, as claimed. □

Theorem 5.9.B

Theorem 5.9.B. The area $\mathcal{A}$ swept out by the line joining the focus F to a point P on the ellipse over an angle φ measured from the semimajor axis (see Figure 5.27, left) is

$$\mathcal{A} = \frac{ab}{2}(u - e \sin u).$$

Proof. We seek the shaded are $\mathcal{A}$ in Figure 5.27 (left).

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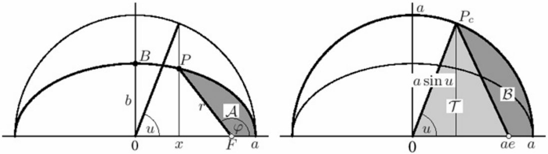

Fig. 5.27. Computation of the area $\mathcal{A}$ swept out by the radius vector

We stretch the ellipse vertically by a factor of a/b (that is, the y-value of each point is multiplied by a/b).

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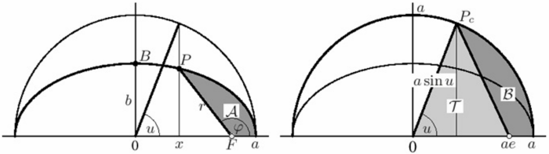

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Proof (continued).

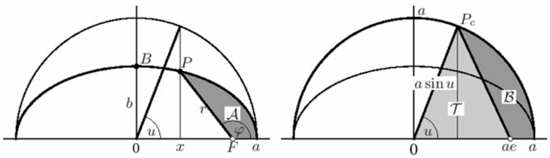

Fig. 5.27. Computation of the area $\mathcal{A}$ swept out by the radius vector

With $\mathcal{B}$ as the shaded area in Figure 5.27 (right) we then have $\mathcal{B} = \frac{a}{b}\mathcal{A}$ (the idea is similar to that of Theorem 1.6, though that does not rigorously justify this claim). The area of the sector in the circle with central angle u measured in radians is $a^2u/2$. With $\mathcal{T}$ as the area of the triangle in Figure 5.27 (right), we have that $a^2u/2$ is then $\mathcal{T} + \mathcal{B}$. Since $\mathcal{T} = \frac{1}{2}(ae)(a\sin u) = \frac{1}{2}a^2 e\sin u$. Therefore,

$$\mathcal{B} = \frac{a^2u}{2} = \mathcal{T} = \frac{a^2u}{2} - \frac{a^2e\sin u}{2} = \frac{a^2}{2}(u - e\sin u), \ldots$$

Theorem 5.9.B (continued 2)

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Proof (continued). ...

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and hence

$$\mathcal{A} = \left(\frac{b}{a}\right) \frac{a^2}{2}(u - e \sin u) = \frac{ab}{2}(u - e \sin u),$$

as claimed. □