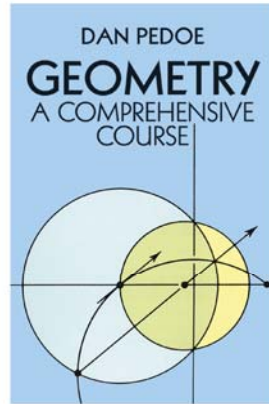


# Real Analysis

## Chapter V. Mappings of the Euclidean Plane 43. The Theorem of Isometries—Proofs of Theorems



## Theorem 43.2. An Auxiliary Theorem

### Theorem 43.2. An Auxiliary Theorem.

Given two pairs of points  $z_0, z_1$  and  $w_0, w_1$  where  $|z_0 - z_1| = |w_0 - w_1| \neq 0$ , there is just one mapping of type  $\mathcal{I}_+$  and just one of type  $\mathcal{I}_-$  which maps  $z_0$  onto  $w_0$  and maps  $z_1$  onto  $w_1$ .

**Proof.** Let  $az + b \in \mathcal{I}_+$  with  $w_0 = az_0 + b$  and  $w_1 = az_1 + b$ . Then  $w_0 - w_1 = (az_0 + b) - (az_1 + b) = a(z_0 - z_1)$  and

$$a = (w_0 - w_1)/(z_0 - z_1)$$

(this is where we use the facts that  $z_0 - z_1 \neq 0$  and  $|z_0 - z_1| = |w_0 - w_1|$ ), so that  $a$  is uniquely determined in terms of the given  $w_0, w_1, z_0, z_1$ . Then

$$b = w_0 - az_0 = w_0 - z_0 \frac{w_0 - w_1}{z_0 - z_1}$$

and  $b$  is uniquely determined (also, ...)

## Theorem 43.2 (continued)

### Proof (continued).

$$\begin{aligned} b &= w_1 - az_1 = w_1 - z_1 \frac{w_0 - w_1}{z_0 - z_1} = \frac{w_1(z_0 - z_1) - z_1(w_0 - w_1)}{z_0 - z_1} \\ &= \frac{w_1z_0 - z_1w_0}{z_0 - z_1} = \frac{z_0w_0 - z_0w_0 + w_1z_0 - z_1w_0}{z_0 - z_1} \\ &= \frac{w_0(z_0 - z_1) - z_0(w_0 - w_1)}{z_0 - z_1} = w_0 - z_0 \frac{w_0 - w_1}{z_0 - z_1}, \end{aligned}$$

as expected).

Similarly, for  $c\bar{z} + d \in \mathcal{I}_-$  with  $w_0 = c\bar{z}_0 + d$  and  $w_1 = c\bar{z}_1 + d$ . Then  $w_0 - w_1 = (c\bar{z}_0 + d) - (c\bar{z}_1 + d) = c(\bar{z}_0 - \bar{z}_1)$  and  $c = (w_0 - w_1)/(\bar{z}_0 - \bar{z}_1)$  so that  $c$  is uniquely determined in terms of the given  $w_0, w_1, z_0, z_1$ . Then  $d = w_0 - c\bar{z}_0 = w_0 - \bar{z}_0(w_0 - w_1)/(\bar{z}_0 - \bar{z}_1)$  and  $d$  is uniquely determined.  $\square$

## Lemma 43.A

**Lemma 43.A.** For distance  $u, v, w \in \mathbb{C}$  we have  $u, v, w$  collinear, with  $v$  between  $u$  and  $w$  on the line containing the points, if and only if  $|v - u| + |w - v| = |w - u|$ .

**Proof.** Suppose  $u, v, w$  are collinear with  $v$  between  $u$  and  $w$ . Say the points lie on the line  $\text{Im}((z - a)/b) = 0$ . The line segment joining  $u$  and  $w$  is  $\{z \mid z = u(1 - t) + tw, t \in [0, 1]\}$ ; notice for such  $z$  we have

$$\begin{aligned} \text{Im} \left( \frac{z - a}{b} \right) &= \text{Im} \left( \frac{(u(1 - t) + tw) - a}{b} \right) \\ &= \text{Im} \left( \frac{u(1 - t) + tw - (1 - t)a - ta}{b} \right) \\ &= \text{Im} \left( \frac{(1 - t)(u - a)}{b} + \frac{t(w - a)}{b} \right) \\ &= (1 - t) \text{Im} \left( \frac{u - a}{b} \right) + t \text{Im} \left( \frac{w - a}{b} \right) = 0 \dots \end{aligned}$$

## Lemma 43.A (continued 1)

**Proof (continued).** since  $u$  and  $w$  are on the line. Since  $v$  is between  $u$  and  $w$  on the line then for some  $t' \in (0, 1)$  we have  $v = u(1 - t') + t'w$ . So  $v - u = u(1 - t') + t'w - u = t'(w - u)$  and  $w - v = w - (u(1 - t') + t'w) = (1 - t')(w - u)$ , and hence

$$\begin{aligned} |v - u| + |w - v| &= |t'(w - u)| + |(1 - t')(w - u)| \\ &= t'|w - u| + (1 - t')|w - u| = |w - u|, \end{aligned}$$

ad claimed.

Now suppose  $|v - u| + |w - v| = w - u|$ . By the Triangle Inequality, this means  $v - u = t''(w - v)$  for some  $t'' \in \mathbb{R}$  and  $t'' > 0$  (notice  $t'' \neq 0$  since  $u, v, w$  are distinct). Then  $t'' = \frac{v - u}{w - v} = \frac{u - v}{v - w}$  and so

$\operatorname{Im}\left(\frac{u - v}{v - w}\right) = \operatorname{Im}(t'') = 0$ . Consider the line in  $\mathbb{C}$   $\operatorname{Im}\left(\frac{z - v}{v - w}\right) = 0$  (the line through  $v$  with "direction vector"  $v - w$ ).

## Lemma 43.A (continued 2)

**Lemma 43.A.** For distance  $u, v, w \in \mathbb{C}$  we have  $u, v, w$  collinear, with  $v$  between  $u$  and  $w$  on the line containing the points, if and only if  $|v - u| + |w - v| = |w - u|$ .

**Proof (continued).** Of course  $\operatorname{Im}\left(\frac{w - v}{v - w}\right) = \operatorname{Im}(-1) = 0$ , so  $u, v$ , and  $w$  are each on this line and  $u, v, w$  are collinear, as claimed. Finally, since  $v - u = t''(w - v)$  then  $(1 + t'')v = u + t''w$  or  $v = \frac{1}{1 + t''}u + \frac{t''}{1 + t''}w$ . As shown above, the line segment joining  $u$  and  $w$  is  $\{z \mid z = (1 - t)u + tw, t \in [0, 1]\}$ , so with  $t = t''/(1 + t'') \in (0, 1)$  and  $1 - t = 1/(1 + t'')$  we see that  $v$  is between  $u$  and  $w$  on this line segment and hence on the line of collinearity of  $u, v$ , and  $w$ , as claimed.  $\square$

## Theorem 43.3. Isometries are Collineations

**Theorem 43.3. Isometries are Collineations.**

Every isometry of the Gauss plane  $\mathbb{C}$  is a collineation.

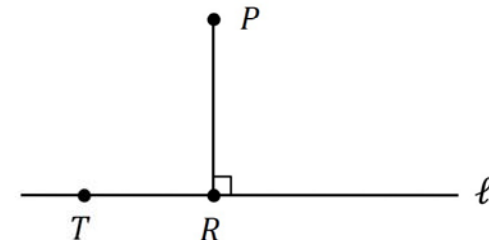
**Proof.** Let  $z \mapsto z'$  be an isometry. Let  $\ell$  be any line in the Gauss plane  $\mathbb{C}$ . Choose three points  $u, v, w$  on  $\ell$  with  $v$  between  $u$  and  $w$  on  $\ell$ . Then by Lemma 43.A,  $|v - u| + |w - v| = |w - u|$ . Since the mapping  $z \mapsto z'$  is an isometry (and we measure distance in  $\mathbb{C}$  using modulus of differences) then  $|v' - u'| + |w' - v'| = |w' - u'|$ . Also by Lemma 43.A,  $u', v', w'$  are collinear (say they lie on line  $\ell'$ ) with  $v'$  between  $u'$  and  $w'$  on the line. That is, the isometry maps line  $\ell$  to line  $\ell'$ . Since  $\ell$  is an arbitrary line in  $\mathbb{C}$ , then the result follows.  $\square$

## Theorem 43.4. Isometries and Parallel Lines

**Theorem 43.4. Isometries and Parallel Lines.**

An isometry of the Gauss plane  $\mathbb{C}$  maps parallel lines onto parallel lines.

**Proof.** If  $P$  is a point in  $\mathbb{C}$  and  $\ell$  is a line that does not contain  $P$ , the minimum distance of  $P$  from line  $\ell$  is well defined (in fact, you could find the point on  $\ell$  which is the minimum distance from  $P$  using Calculus 1), and given by  $PR$  where  $R$  is the intersection of the perpendicular to  $\ell$  through  $P$  with line  $\ell$ .



## Theorem 43.4 (continued 1)

**Proof.** We can establish this using Euclid by noticing that any other point  $T$  on line  $\ell$  produces a right triangle  $PRT$  and by the Pythagorean Theorem (Euclid Book I, Proposition 47), say,  $PR$  is smaller than the hypotenuse  $PT$ . Under an isometry, line  $\ell$  containing point  $P$  is mapped to some line  $\ell'$  containing  $P'$  (by Theorem 43.3), and line  $PR$  is mapped to line  $P'R'$ . Since the mapping is an isometry, then the minimum distance from line  $\ell'$  to point  $P'$  is given by  $P'R'$  so that line  $P'R'$  is perpendicular to line  $\ell'$  (this shows that isometries map perpendicular lines to perpendicular lines).

Given two parallel lines  $\ell$  and  $m$ , take two points  $P$  and  $Q$  on  $m$  and let points  $R$  and  $S$  be the intersections of perpendicular to  $\ell$  through these points (respectively) with line  $\ell$ . Since parallel lines are equidistant from each other then we have the equality  $PR = QS$  of distances.

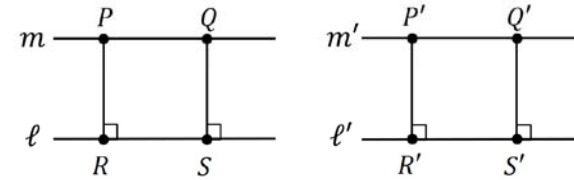
## Lemma 43.B

**Lemma 43.B.** If three circles with different centers intersect in two points then the centers of the circles must be collinear.

**Proof.** Let the circles be  $C_i$  with centers  $z_i$  (respectively) for  $i = 0, 1, 2$ . Suppose the distinct points  $z$  and  $w$  lie on all three circles. Consider the line segment  $zw$ . If  $zw$  is a diameter of one of the circles then construct the line  $\ell$  perpendicular to  $zw$  and passing through the center of that circle (we use Euclid's Book I, Proposition 11: "To draw a straight line at right angles to a given straight line from a given point on it."). Otherwise, let  $v$  be the midpoint of  $zw$  (we use Book I, Proposition 10: "To bisect a given finite straight line.") and construct the line  $\ell$  through  $v$  and the center of one of the circles. Euclid's Book III, Proposition 3 state: "If a straight line passing through the center of a circle bisects a straight line not passing through the center, then it also cuts it at right angles; and if it cuts it at right angles, then it also bisects it." By Book III, Proposition 3, line  $\ell$  is perpendicular to segment  $zw$ .

## Theorem 43.4 (continued 2)

**Proof.**



With lines  $\ell'$  and  $m'$  as the images of lines  $\ell$  and  $m$  respectively and with points  $P', Q', R', S'$  as the images of points  $P, Q, R, S$ , respectively, we have that lines  $P'R'$  and  $Q'S'$  are both perpendicular to line  $\ell'$  as argued above. Also, since the mapping is an isometry, we have the equality of distances  $P'R' = Q'S'$ . So lines  $\ell'$  and  $m'$  are equidistant from each other and hence are parallel (since the distance is greater than 0 then the lines do not intersect; that is, they are parallel). Since parallel lines  $\ell$  and  $m$  are arbitrary, the claim follows.  $\square$

## Lemma 43.B (continued)

**Lemma 43.B.** If three circles with different centers intersect in two points then the centers of the circles must be collinear.

**Proof (continued).** Euclid's Book III, Corollary to Proposition 1 state: "If in a circle a straight line cuts a straight line into two equal parts and at right angles, then the center of the circle lies on the cutting straight line." Now the other two circles also both contain points  $z$  and  $w$  so by Book III, Corollary to Proposition 1, since  $\ell$  bisects segment  $zw$  and is perpendicular to it, the  $\ell$  contains the center of the other two circles as well, as claimed.  $\square$

## Theorem 43.5. Determination of an Isometry

**Theorem 43.5. Determination of an Isometry.**

An isometry of the Gauss plane  $\mathbb{C}$  is uniquely determined by the assignment of the congruent maps of a given triangle. That is, if  $z_0, z_1, z_2$  are noncollinear points with respective images  $w_0, w_1, w_2$  then for any  $z$  in the plane, the image of  $z$  is uniquely determined from  $w_0, w_1, w_2$ .

**Proof.** Let  $z_0, z_1, z_2$  be noncollinear points in the Gauss plane  $\mathbb{C}$ . By Lemma 43.A we have:  $|z_0 - z_1| + |z_1 - z_2| < |z_0 - z_2|$ ,  $|z_0 - z_1| + |z_0 - z_2| < |z_1 - z_2|$ , and  $|z_0 - z_2| + |z_1 - z_2| < |z_0 - z_1|$  since the points are noncollinear and equality in any one of these three would imply linearity of the three points. Now  $|z_0 - z_2| = |w_0 - w_1|$ ,  $|z_1 - z_2| = |w_1 - w_2|$ , and  $|z_0 - z_2| = |w_0 - w_2|$  since we have an isometry. So  $|w_0 - w_1| + |w_1 - w_2| < |w_0 - w_2|$ ,  $|w_0 - w_1| + |w_0 - w_2| < |w_1 - w_2|$ , and  $|w_0 - w_2| + |w_1 - w_2| < |w_0 - w_1|$  and the points  $w_0, w_1, w_2$  are not collinear.

## Theorem 43.5 (continued)

**Proof (continued).** Let  $z$  be a point  $\mathbb{C}$  other than  $z_0, z_1, z_2$ . Consider the circles  $C_i$  with (respective) centers  $z_i$  and radius  $|z - z_i|$  for  $i = 0, 1, 2$ . Then the three circles intersect at point  $z$ . Since the centers are not collinear, then by Lemma 43.B  $z$  is the only point on the three circles. That is, point  $z$  is uniquely determined by the three distances  $|z - z_0|$ ,  $|z - z_1|$ , and  $|z - z_2|$ . Now triangle  $w_0w_1w_2$  is congruent to triangle  $z_0z_1z_2$  and similarly there is a *unique* point on the intersection of the three circles  $C'_i$  centered at  $w_i$  with radii  $|z - z_i|$  for  $i = 0, 1, 2$ ; denote the unique point as  $w$ . Since the mapping is an isometry then we must have  $w$  as the image of  $z$ . Since  $z$  is an arbitrary point in  $\mathbb{C}$  (distinct from  $z_0, z_1, z_2$ ) then the isometry on  $\mathbb{C}$  is uniquely determined.  $\square$

## Theorem 43.6

**Theorem 43.6.** There are precisely two isometries of the Gauss plane which map two given points  $z_0$  and  $z_1$  into two given points  $w_0$  and  $w_1$  (respectively) where  $|z_0 - z_1| = |w_0 - w_1| \neq 0$ .

**Proof.** Theorem 43.2 gives two such isometries, one in  $\mathcal{I}_+$  and one in  $\mathcal{I}_-$ . We now show that these are the only such isometries. Let  $z$  be a point in  $\mathbb{C}$  that is not collinear with  $z_0$  and  $z_1$ . Consider circle  $C_0$  centered at  $z_0$  with radius  $|z - z_0|$  and circle  $C_1$  centered at  $z_1$  with radius  $|z - z_1|$ . Since  $z$  lies on both  $C_0$  and  $C_1$  and  $z$  is not collinear with the centers of  $z_0$  and  $z_1$  then these circles intersect at two points. Similarly, circle  $C'_0$  centered at  $w_0$  with radius  $|z - z_0|$  and circle  $C'_1$  centered at  $w_1$  with radius  $|z - z_1|$  intersect in two points. Since the mapping is an isometry, then  $z$  must be mapped to either one or the other of the two points on circles  $C'_0$  and  $C'_1$ . So there are at most two such isometries, and the result follows.  $\square$

## Theorem 43.1. The Main Theorem on Isometries of the Gauss Plane

**Theorem 43.1. The Main Theorem on Isometries of the Gauss Plane.**

The set  $\mathcal{I}$  of all isometries of the Gauss plane  $\mathbb{C}$  (onto itself) is composed of two classes  $\mathcal{I}_+$  and  $\mathcal{I}_-$ . The class  $\mathcal{I}_+$  consists of all isometries of the form  $z' = az + b$  where  $|a| = 1$ , and the class  $\mathcal{I}_-$  of all isometries of the form  $z' = c\bar{z} + d$  where  $|c| = 1$ .

**Proof.** Consider a given isometry of the Gauss plane  $\mathbb{C}$ . Let  $z_0$  and  $z_1$  be any distinct points in  $\mathbb{C}$  with images  $w_0$  and  $w_1$ , respectively, under the isometry. By Theorem 43.2, there are two possibilities for the isometry, one in  $\mathcal{I}_+$  and one in  $\mathcal{I}_-$ . By Theorem 43.6, there are only two possibilities for the isometry. So the isometry must be in either  $\mathcal{I}_+$  or  $\mathcal{I}_-$  and hence every isometry of  $\mathbb{C}$  is contained in either  $\mathcal{I}_+$  or  $\mathcal{I}_-$ , as claimed.  $\square$