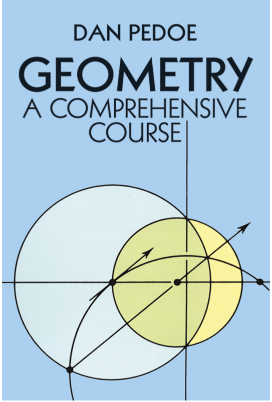
Real Analysis

Chapter V. Mappings of the Euclidean Plane

43. The Theorem of Isometries—Proofs of Theorems



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Proof. Let $az + b \in \mathscr{I}_+$ with $w_0 = az_0 + b$ and $w_1 = az_1 + b$. Then $w_0 - w_1 = (az_0 + b) - (az_1 + b) = a(z_0 - z_1)$ and

$$a = (w_0 - w_1)/(z_0 - z_1)$$

(this is where we use the facts that $z_0 - z_1 \neq 0$ and $|z_0 - z_1| = |w_0 - w_1|$), so that a is uniquely determined in terms of the given w_0, w_1, z_0, z_1.

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Similarly, for $c\overline{z} + d \in \mathscr{I}_-$ with $w_0 = c\overline{z}_0 + d$ and $w_1 = c\overline{z}_1 + d$. Then $w_0 - w_1 = (c\overline{z}_0 + d) - (c\overline{z}_1 + d) = c(\overline{z}_0 - \overline{z}_1)$ and $c = (w_0 - w_1)/(\overline{z}_0 - \overline{z}_1)$ so that c is uniquely determined in terms of the given w_0, w_1, z_0, z_1. Then $d = w_0 - c\overline{z}_0 = w_0 - \overline{z}_0(w_0 - w_1)/(\overline{z}_0 - \overline{z}_1)$ and d is uniquely determined. □

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Lemma 43.A

Lemma 43.A. For distance $u, v, w \in \mathbb{C}$ we have u, v, w collinear, with v between u and w on the line containing the points, if and only if $|v-u| + |w-v| = |w-u|$.

Proof. Suppose u, v, w are collinear with v between u and w. Say the points lie on the line $\mathrm{Im}((z-a)/b) = 0$. The line segment joining u and w is $\{z \mid z = u(1-t) + tw, t \in [0,1]\}$; notice for such z we have

$$\begin{aligned}
\mathrm{Im}\left(\frac{z-a}{b}\right) &= \mathrm{Im}\left(\frac{(u(1-t)+tw)-a}{b}\right) \\
&= \mathrm{Im}\left(\frac{u(1-t)+tw-(1-t)a-ta}{b}\right) \\
&= \mathrm{Im}\left(\frac{(1-t)(u-a)}{b} + \frac{t(w-a)}{b}\right) \\
&= (1-t)\mathrm{Im}\left(\frac{u-a}{b}\right) + t\mathrm{Im}\left(\frac{w-a}{b}\right) = 0\ldots
\end{aligned}$$

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Proof (continued). since u and w are on the line. Since v is between u and w on the line then for some $t' \in (0,1)$ we have $v = u(1-t') + t'w$. So $v - u = u(1-t') + t'w - u = t'(w-u)$ and $m - v = w - (u(1-t') + t'w) = (1-t')(w-u)$, and hence

$$|v-u| + |w-v| = |t'(w-u)| + |(1-t')(w-u)|$$

$$= t'|w-u| + (1-t')|w-u| = |w-u|,$$

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Now suppose $|v-u| + |w-v| = w-u|$. By the Triangle Inequality, this means $v - u = t''(w-v)$ for some $t'' \in \mathbb{R}$ and $t'' > 0$ (notice $t'' \neq 0$ since u, v, w are distinct). Then $t'' = \dfrac{v-u}{w-v} = \dfrac{u-v}{v-w}$ and so $\operatorname{Im}\left(\dfrac{u-v}{v-w}\right) = \operatorname{Im}(t'') = 0$.

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$\mathrm{Im}\left(\dfrac{u-v}{v-w}\right) = \mathrm{Im}(t'') = 0$. Consider the line in $\mathbb{C}$ $\mathrm{Im}\left(\dfrac{z-v}{v-w}\right) = 0$ (the line through v with "direction vector" $v-w$).

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Every isometry of the Gauss plane $\mathbb{C}$ is a collineation.

Proof. Let $z \mapsto z'$ be an isometry. Let ℓ be any line in the Gauss plane $\mathbb{C}$ Choose three points u, v, w on ℓ with v between u and w on ℓ. Then by Lemma 43.A, $|v - u| + |w - v| = |w - u|$. Since the mapping $z \mapsto z'$ is an isometry (and we measure distance in $\mathbb{C}$ using modulus of differences) then $|v' - u'| + |w' - v'| = |w' - u'|$.

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Theorem 43.4. Isometries and Parallel Lines

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An isometry of the Gauss plane $\mathbb{C}$ maps parallel lines onto parallel lines.

Proof. If P is a point in $\mathbb{C}$ and ℓ is a line that does not contain P, the minimum distance of P from line ℓ is well defined (in fact, you could find the point on ℓ which is the minimum distance from P using Calculus 1), and given by PR where R is the intersection of the perpendicular to ℓ through P with line ℓ.

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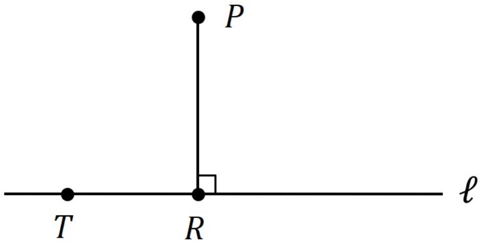

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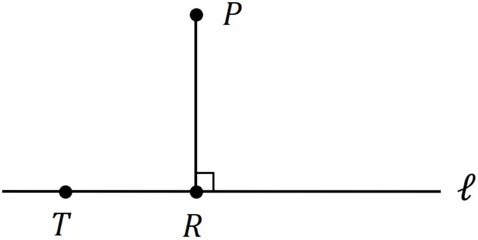

Theorem 43.4 (continued 1)

Proof. We can establish this using Euclid by noticing that any other point T on line ℓ produces a right triangle PRT and by the Pythagorean Theorem (Euclid Book I, Proposition 47), say, PR is smaller than the hypotenuse PT. Under an isometry, line ℓ containing point P is mapped to some line ℓ' containing P' (by Theorem 43.3), and line PR is mapped to line $P'R'$. Since the mapping is an isometry, then the minimum distance from line ℓ' to point P' is given by $P'R'$ so that line $P'R'$ is perpendicular to line ℓ' (this shows that isometries map perpendicular lines to perpendicular lines).

Given two parallel lines ℓ and m, take two points P and Q on m and let points R and S be the intersections of perpendicular to ℓ through these points (respectively) with line ℓ. Since parallel lines are equidistant from each other then we have the equality $PR = QS$ of distances.

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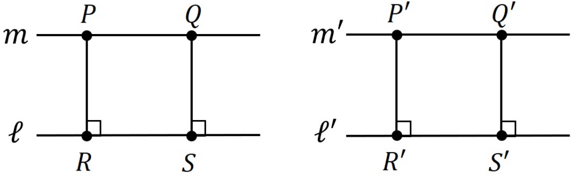
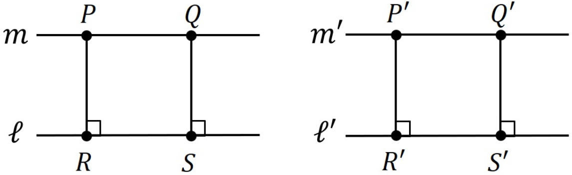


With lines ℓ' and m' as the images of lines ℓ and m respectively and with points P', Q', R', S' as the images of points P, Q, R, S, respectively, we have that lines $P'R'$ and $Q'R'$ are both perpendicular to line ℓ' as argued above. Also, since the mapping is an isometry, we have the equality of distances $P'R' = Q'S'$. So lines ℓ' and m' are equidistant from each other and hence are parallel (since the distance is greater than 0 then the lines do not intersect; that is, they are parallel). Since parallel lines ℓ and m are arbitrary, the claim follows. □

Theorem 43.4 (continued 2)

Proof.



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Lemma 43.B

Lemma 43.B. If three circles with different centers intersect in two points then the centers of the circles must be collinear.

Proof. Let the circles be C_i with centers z_i (respectively) for $i = 0, 1, 2$. Suppose the distinct points z and w lie on all three circles. Consider the line segment zw.

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Proof. Let the circles be C_i with centers z_i (respectively) for $i = 0, 1, 2$. Suppose the distinct points z and w lie on all three circles. Consider the line segment zw. If zw is a diameter of one of the circles then construct the line ℓ perpendicular to zw and passing through the center of that circle (we use Euclid's Book I, Proposition 11: "To draw a straight line at right angles to a given straight line from a given point on it."). Otherwise, let v be the midpoint of zw (we use Book I, Proposition 10: "To bisect a given finite straight line.") and construct the line ℓ through v and the center of one of the circles.

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Lemma 43.B (continued)

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Proof (continued). Euclid's Book III, Corollary to Proposition 1 state: "If in a circle a straight line cuts a straight line into two equal parts and at right angles, then the center of the circle lies on the cutting straight line." Now the other two circles also both contain points z and w so by Book III, Corollary to Proposition 1, since ℓ bisects segment zw and is perpendicular to it, the ℓ contains the center of the other two cycles as well, as claimed. □

Theorem 43.5. Determination of an Isometry

Theorem 43.5. Determination of an Isometry.
An isometry of the Gauss plane $\mathbb{C}$ is uniquely determined by the assignment of the congruent maps of a given triangle. That is, if z_0, z_1, z_2 are noncollinear points with respective images w_0, w_1, w_2 then for any z in the plane, the image of z is uniquely determined from w_0, w_1, w_2.

Proof. Let z_0, z_1, z_2 be noncollinear points in the Gauss plane $\mathbb{C}$. By Lemma 43.A we have: $|z_0 - z_1| + |z_1 - z_2| < |z_0 - z_2|$, $|z_0 - z_1| + |z_0 - z_2| < |z_1 - z_2|$, and $|z_0 - z_2| + |z_1 - z_2| < |z_0 - z_1|$ since the points are noncollinear and equality in any one of these three would imply linearity of the three points. Now $|z_0 - z_2| = |w_0 - w_1|$, $|z_1 - z_2| = |w_1 - w_2|$, and $|z_0 - z_2| = |w_0 - w_2|$ since we have an isometry. So $|w_0 - w_1| + |w_1 - w_2| < |w_0 - w_2|$, $|w_0 - w_1| + |w_0 - w_2| < |w_1 - w_2|$, and $|w_0 - w_2| + |w_1 - w_2| < |w_0 - w_1|$ and the points w_0, w_1, w_2 are not collinear.

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Theorem 43.5 (continued)

Proof (continued). Let z be a point $\mathbb{C}$ other than z_0, z_1, z_2. Consider the circles C_i with (respective) centers z_i and radius $|z - z_i|$ for $i = 0, 1, 2$. Then the three circles intersect at point z. Since the centers are not collinear, then by Lemma 43.B z is the only point on the three circles. That is, point z is uniquely determined by the three distances $|z - z_0|$, $|z - z_1|$, and $|z - z_2|$. Now triangle $w_0 w_1 w_2$ is congruent to triangle $z_0 z_1 z_2$ and similarly there is a *unique* point on the intersection of the three circles C_i' centered at w_i with radii $|z - z_i|$ for $i = 0, 1, 2$; denote the unique point as w. Since the mapping is an isometry then we must have w as the image of z. Since z is an arbitrary point in $\mathbb{C}$ (distinct from z_0, z_1, z_2) then the isometry on $\mathbb{C}$ is uniquely determined. □

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Theorem 43.6

Theorem 43.6. There are precisely two isometries of the Gauss plane which map two given points z_0 and z_1 into two given points w_0 and w_1 (respectively) where $|z_0 - z_1| = |w_0 - w_1| \neq 0$.

Proof. Theorem 43.2 gives two such isometries, one in $\mathscr{I}_+$ and one in $\mathscr{I}_-$. We now show that these are the only such isometries. Let z be a point in $\mathbb{C}$ that is not collinear with z_0 and z_1. Consider circle C_0 centered at z_0 with radius $|z - z_0|$ and circle C_1 centered at z_1 with radius $|z - z_1|$. Since z lies on both C_0 and C_1 and z is not collinear with the centers of z_0 and z_1 then these circles intersect at two points.

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Theorem 43.1. The Main Theorem on Isometries of the Gauss Plane

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The set $\mathscr{I}$ of all isometries of the Gauss plane $\mathbb{C}$ (onto itself) is composed of two classes $\mathscr{I}_+$ and $\mathscr{I}_-$. The class $\mathscr{I}_+$ consists of all isometries of the form $z' = az + b$ where $|a| = 1$, and the class $\mathscr{I}_-$ of all isometries of the form $z' = c\overline{z} + d$ where $|c| = 1$.

Proof. Consider a given isometry of the Gauss plane $\mathbb{C}$. Let z_0 and z_1 be any distinct points in $\mathbb{C}$ with images w_0 and w_1, respectively, under the isometry. By Theorem 43.2, there are two possibilities for the isometry, one in $\mathscr{I}_+$ and one in $\mathscr{I}_-$.

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