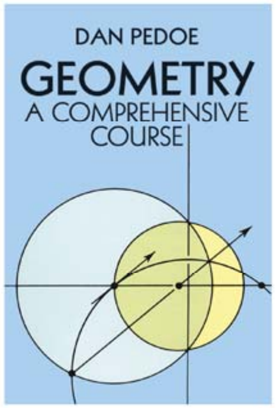
Real Analysis

Chapter V. Mappings of the Euclidean Plane

44. Algebra and Groups—Proofs of Theorems



Theorem 44.2. Conditions for a Subgroup

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A nonempty subset H of a group G is a subgroup of G if and only if for every $a, b \in H$ we have (i) $b^{-1}, ab \in H$, or (ii) $ab^{-1} \in H$.

Proof. First, if H is a subgroup and $a, b \in H$ then, since H itself is a group, $b^{-1} \in H$ by the Inverse Law, and so by closure under the binary operation, $ab, ab^{-1} \in H$.

Second, suppose for all $a, b \in H$ that (ii) holds and so $ab^{-1} \in H$. Then for all $a \in H$ (with $b = a$) we have $aa^{-1} = i \in H$ so H satisfies The Identity Law. So for all $b \in H$ (with $a = i$) we have $ib^{-1} = b^{-1} \in H$ and so H satisfies The Inverse Law. The Associative Law is satisfies on G and so is satisfies on a subset of G. Therefore, (ii) implies that H is a subgroup of G. Next, suppose (i) holds and that for $a, b \in H$ we have $b^{-1}, ab \in H$. So both $b, b^{-1} \in H$ and hence $ab^{-1} \in H$ and (i) implies (ii). Since (ii) implies that H is a subgroup of G, then (i) implies that H is a subgroup of G. □

Theorem 44.4. The Identity of Cosets

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If Ha and Hb have one element in common then they coincide (that is, they are equal).

Proof. Let $c \in Ha \cap Hb$. Then $c = ha = kb$ for some $h, k \in H$. From $ha = kb$ we have $h^{-1}k = ab^{-1} \in H$. We know $h^{-1}k \in H$ since H is a group and since $ha = kb$ then $h^{-1}k = ab^{-1} \in H$. Also, $a = (h^{-1}k)b \in Hb$. So for any $h'a \in Ha$ we have $h'a = h'(h^{-1}k)b \in Hb$ so that $Ha \subset Hb$. Similarly, $b = (k^{-1}h)a \in Ha$ and for any $k'b \in Hb$ we have $k'(k^{-1}h)a \in Ha$ so that $Hb \subset Ha$. So $Ha = Hb$, as claimed. □

Corollary 44.4

Corollary 44.4. Elements $a, b \in G$ lie in the same right coset of H if and only if $ab^{-1} \in H$.

Proof. Let $g \in G$ and H a subgroup of G. Since the identity $i \in H$ then $g \in Hg$ so that every element of G lies in some right coset. (Notice that this, combined with Theorem 44.4, implies that the cosets of H partition G.) Suppose a and b lie in the same right coset of H. Since $a \in Ha$ and $b \in Gb$ then we have $Ha = Hb$ and, as shown in the proof of Theorem 44.4, $ab^{-1} \in H$, as claimed.

On the other hand, if $ab^{-1} \in H$, say $ab^{-1} = h \in H$, then $a = hb \in Hb$ so that a is in both Ha and Hb. By Theorem 44.4, $Ha = Hb$ and so a and b lie in the same right coset. □

Theorem 44.5. Right and Left Cosets

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If the number of right cosets with respect to a subgroup H is finite, then there is an equal number of left cosets, and conversely.

Proof. If $h \in H$ then $(ah)^{-1} = h^{-1}a^{-1}$. Since H is a group, as h^{-1} "runs through" all the elements of H, then h correspondingly "runs through" all elements of H (that is, $h \in H$ if and only if $h^{-1} \in H$). So $\{h^{-1}a^{-1} \mid h^{-1} \in H\} = \{ha^{-1} \mid h \in H\} = Ha^{-1}$. Hence we can associate each left coset aH with the right coset Ha^{-1}. This association is onto (since any right coset Hb is the image of $b^{-1}H$ under this association). By Corollary 44.4, $Ha^{-1} = Hb^{-1}$ if and only if $(a^{-1})(b^{-1})^{-1} = a^{-1}b \in H$, and so $(a^{-1}b)^{-1} = b^{-1}a \in H$. By Note 44.A, this implies the left cosets aH and bH are equal and so the association is one to one. Therefore, the mapping of left cosets to right cosets given by $aH \to Ha$ (for all $a \in G$) is a bijection (or a "one-to-one correspondence") and so the number of left cosets equals the number of right cosets. □

Theorem 44.A. Lagrange's Theorem

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If G is a finite group and H is a subgroup of G then the order of H divides the order of G.

Proof. By Exercise 44.2, all right cosets of H are of the order of H. The index $[G : H]$ is the number of right cosets of H. By Note 44.B, the cosets of H in G partition G. So $|G| = [G : H]|H|$. So the order of H divides the order of G (namely, $[G : H]$ times). □