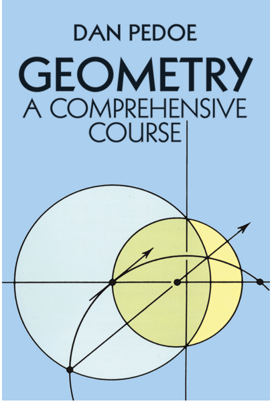
Real Analysis

Chapter V. Mappings of the Euclidean Plane

44. Algebra and Groups—Proofs of Theorems



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Theorem 44.2. Conditions for a Subgroup

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A nonempty subset H of a group G is a subgroup of G if and only if for every $a, b \in H$ we have (i) $b^{-1}, ab \in H$, or (ii) $ab^{-1} \in H$.

Proof. First, if H is a subgroup and $a, b \in H$ then, since H itself is a group, $b^{-1} \in H$ by the Inverse Law, and so by closure under the binary operation, $ab, ab^{-1} \in H$.

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Second, suppose for all $a, b \in H$ that (ii) holds and so $ab^{-1} \in H$. Then for all $a \in H$ (with $b = a$) we have $aa^{-1} = i \in H$ so H satisfies The Identity Law. So for all $b \in H$ (with $a = i$) we have $ib^{-1} = b^{-1} \in H$ and so H satisfies The Inverse Law. The Associative Law is satisfies on G and so is satisfies on a subset of G. Therefore, (ii) implies that H is a subgroup of G.

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Theorem 44.4. The Identity of Cosets

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If Ha and Hb have one element in common then they coincide (that is, they are equal).

Proof. Let $c \in Ha \cap Hb$. Then $c = ha = kb$ for some $h, k \in H$. From $ha = kb$ we have $h^{-1}k = ab^{-1} \in H$. We know $h^{-1}k \in H$ since H is a group and since $ha = kb$ then $h^{-1}k = ab^{-1} \in H$. Also, $a = (h^{-1}k)b \in Hb$. So for any $h'a \in Ha$ we have $h'a = h'(h^{-1}k)b \in Hb$ so that $Ha \subset Hb$.

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Corollary 44.4

Corollary 44.4. Elements $a, b \in G$ lie in the same right coset of H if and only if $ab^{-1} \in H$.

Proof. Let $g \in G$ and H a subgroup of G. Since the identity $i \in H$ then $g \in Hg$ so that every element of G lies in some right coset. (Notice that this, combined with Theorem 44.4, implies that the cosets of H partition G.) Suppose a and b lie in the same right coset of H. Since $a \in Ha$ and $b \in Gb$ then we have $Ha = Hb$ and, as shown in the proof of Theorem 44.4, $ab^{-1} \in H$, as claimed.

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On the other hand, if $ab^{-1} \in H$, say $ab^{-1} = h \in H$, then $a = hb \in Hb$ so that a is in both Ha and Hb. By Theorem 44.4, $Ha = Hb$ and so a and b lie in the same right coset. □

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Theorem 44.5. Right and Left Cosets

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If the number of right cosets with respect to a subgroup H is finite, then there is an equal number of left cosets, and conversely.

Proof. If $h \in H$ then $(ah)^{-1} = h^{-1}a^{-1}$. Since H is a group, as h^{-1} "runs through" all the elements of H, then h correspondingly "runs through" all elements of H (that is, $h \in H$ if and only if $h^{-1} \in H$). So $\{h^{-1}a^{-1} \mid h^{-1} \in H\} = \{ha^{-1} \mid h \in H\} = Ha^{-1}$. Hence we can associate each left coset aH with the right coset Ha^{-1}.

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Theorem 44.A. Lagrange's Theorem

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If G is a finite group and H is a subgroup of G then the order of H divides the order of G.

Proof. By Exercise 44.2, all right cosets of H are of the order of H. The index $[G:H]$ is the number of right cosets of H. By Note 44.B, the cosets of H in G partition G. So $|G| = [G:H]|H|$. So the order of H divides the order of G (namely, $[G:H]$ times). □

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