Real Analysis

Chapter V. Mappings of the Euclidean Plane

45. Conjugate and Normal Subgroups—Proofs of Theorems

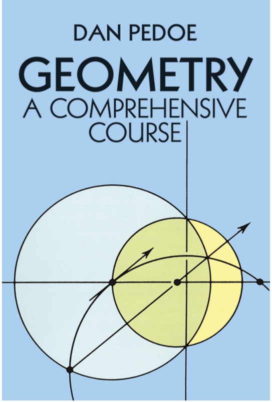

Table of contents

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Proof. If $h, k \in H$ then $g^{-1}kg, g^{-1}kg \in K$ and to show that K is a subgroup of G, by Theorem 44.2 (since ^{-1}kg and $g^{-1}hg$ are arbitrary elements of K) it is sufficient to show that $(g^{-1}kg)(g^{-1}hg)^{-1} \in K$. Now

$$(g^{-1}kg)(g^{-1}hg)^{-1} = g^{-1}kgg^{-1}h^{-1}g = g^{-1}kh^{-1}g \in K$$

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Theorem 45.2. A Classification of Normal Subgroups

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H is a normal subgroup of G if and only if $h \in H$ implies $g^{-1}hg \in H$ for all $g \in G$.

Proof. Suppose $g^{-1}hg \in H$ for all $h \in H$ and for all $g \in G$. Then $g^{-1}Hg \subset H$ for all $g \in G$. Since $g^{-1} \in G$ we also have $(g^{-1})^{-1}Hg^{-1} = gHg^{-1} \subset H$ or $H \subset g^{-1}Hg$. Therefore $H = g^{-1}Hg$ and so (by definition) H is a normal subgroup of G.

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Theorem 45.3. Subgroups of Index Two

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A subgroup H of index two in G is always normal.

Proof. We use Note 45.A (based on Theorem 45.2). For any $h \in H$ we have $hH = Hh = H$ (or $h^{-1}Hh = H$). If $g \in G$ and $g \notin H$ then the right coset Hg contains g and so $Hg \neq H$ so Hg must be the other right coset of H (namely, $G \setminus H$).

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Proof. Since $\alpha(x) = g^{-1}xg$, then for any $h \in G$ we have $x = ghg^{-1} \in G$ and $\alpha(x) = \alpha(ghg^{-1}) = g^{-1}(ghg^{-1})g = h$ and so α is onto. If $\alpha(x) = \alpha(y)$ then $g^{-1}xg = g^{-1}yg$ and so $x = y$. That is, α if one to one. So α is a bijection.

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Finally, for all $a, b \in G$,

$$\alpha(ab) = g^{-1}(ab)g = g^{-1}agg^{-1}bg = (g^{-1}ag)(g^{-1}bg) = \alpha(a)\alpha(b).$$

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