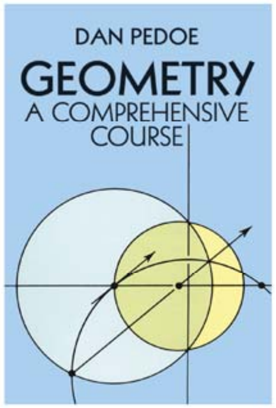
Real Analysis

Chapter V. Mappings of the Euclidean Plane

46. Groups of Mappings—Proofs of Theorems



Theorem 46.1

Theorem 46.1. The translation of the Gauss plane $\mathbb{C}$ form an Abelian group $\mathscr{T}$ which is isomorphic to the additive group of complex numbers. The group operation on $\mathscr{T}$ is composition of mappings.

Proof. Let $t_b : z' = z + b$, $T_c : z' = z + c$, and $T_d : z' = z + d$ be translations. Then $t_b \circ T_c : z' = (z + b) + c = z + (c + b)$ and so composition actually is a binary operation on $\mathscr{T}$. Also $T_c \circ T_c : z' = (z + b) + c = z + (b + c)$, so $T_b \circ T_c = T_b \circ T_c$ and the binary operation is commutative. Now function composition is associative in general, and we have here specifically $T_b \circ (T_c \circ T_d) : z' = (z + (d + c)) + b = z + (d + c + b)$ and $(T_b \circ T_c) \circ T_d : z' = (z + d) + (c + b) = z + (d + c + b)$ so that $T_b \circ (T_c \circ T_d) = (T_b \circ T_c) \circ T_d$ and The Associative Law holds. The identity is $T_0 : z' = z + 0 = z$ and the inverse of $T_b : z' = z + b$ is $(T_b)^{-1} = T_{-b} : z' = z - b$, so that The Law of Identity and The Law of Inverse hold. So $\mathscr{T}$ is a group with a commutative binary operation. That is, $\mathscr{T}$ is an Abelian group, as claimed.

Theorem 46.1 (continued)

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Proof (continued). To prove the isomorphism, map $T_b \in \mathscr{T}$ to $b \in \mathbb{C}$. The mapping is "clearly" one to one and onto. For $T_b, T_c \in \mathscr{T}$, the mapping sends $T_b \circ T_c$ to $b + c$ (since $T_b \circ T_c : z + (b + c)$), so the mapping is an isomorphism, as claimed. □

Theorem 46.2. The Group of Dilative Rotations

Theorem 46.2. The Group of Dilative Rotations. The dilative rotations of the Gauss plane $\mathbb{C}$ about the origin (of the form $z' = az$) form an Abelian group $\mathscr{D}$ (where the binary operation is composition) which is isomorphic to the multiplicative group of the non-zero complex numbers.

Proof. For every non-zero $a \in \mathbb{C}$, define $D_a : z' = az$. First, $D_a \circ D_b : z' = a(bz) = (ab)z$ so composition really is a binary operation on $\mathscr{D}$. Also, $D_b \circ D_a : z' = b(az) = (ba)z = (ab)z$ so that the binary operation is commutative. The identity is $D_1 : z' = z$ and The Identity Law holds. The inverse of D_a is $D_{z^{-1}}$ since $D_a \circ D_{a^{-1}} : z' = a(a^{-1}z) = (aa^{-1})z = z$ and The Inverse Law holds. Finally, $D_a \circ (D_b \circ D_c) : z' = a(bcz) = (abc)z$ and $(D_a \circ (D_b \circ D_c) : z' = (ab)(cz) = (abc)z$ so $D_a \circ (D_b \circ D_c) = (D_a \circ D_b) \circ D_c$ so that The Associative Law holds and so $\mathscr{D}$ is an Abelian group.

Theorem 46.2 (continued)

Theorem 46.2. The Group of Dilative Rotations.
The dilative rotations of the Gauss plane $\mathbb{C}$ about the origin (of the form $z' = az$) form an Abelian group $\mathscr{D}$ (where the binary operation is composition) which is isomorphic to the multiplicative group of the non-zero complex numbers.

Proof (continued). If we map $D_a \mapsto z$ then the mapping is "clearly" a bijection between $\mathscr{D}$ and $\mathbb{C} \setminus \{0\}$. Since $D_1 \circ D_b : z' = (ab)z$ and $D_{ab} : z' = (ab)z$ then the mapping of $D_z \circ D_b$ is the same as the mapping of $D_a \circ D_b$ is the same as the mapping of D_{ab} (since both are mapped to ab). Therefore the map is an isomorphism between $\mathscr{D}$ and $\mathbb{C} \setminus \{0\}$ under multiplication, as claimed. □

Theorem 46.3. Groups of Central Dilations and Rotations

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The central dilations (of the form $z' = az$ where $a \in \mathbb{R}$ and $a \neq 0$) form a subgroup $\mathscr{D}^*$ of $\mathscr{D}$. The rotations about the origin (of the form $z' = az$ where $|a| = 1$) form a subgroup $\mathscr{R}_0$ of $\mathscr{D}$. Both $\mathscr{D}^*$ and $\mathscr{R}_0$ are Abelian.

Proof. For central dilations, consider D_a and $(D_b)^{-1}$ in $\mathscr{D}^*$. We see in the proof of Theorem 46.2 that $D_b^{-1} = D_{b^{-1}}$ and so $D_a \circ (D_b)^{-1} = D_a \circ D_{b^{-1}} : z' = a(b^{-1}z) = (ab^{-1})z$. Since $a, b \in \mathbb{R}$ (and are non-zero) then $ab^{-1} \in \mathbb{R}$ and so $D_a \circ (D_b)^{-1} = D_{ab^{-1}} \in \mathscr{D}^*$ and by Theorem 44.2, $\mathscr{D}^*$ is a subgroup of $\mathscr{D}$.

For rotations about the origin, consider D_a and $(D_b)^{-1}$ in $\mathscr{R}_0$ (so that $|a| = |b| = 1$). Then, as above, $D_a \circ (D_b)^{-1} = D_{ab^{-1}}$ and since $|ab^{-1}| = |a||b^{-1}| = 1$ then $D_a \circ (D_b)^{-1} \in \mathscr{R}_0$ and by Theorem 44.2, $\mathscr{R}_0$ is a subgroup of $\mathscr{D}$.

Since $\mathscr{D}$ is Abelian then $\mathscr{D}^*$ and $\mathscr{R}_0$ are Abelian. □

Theorem 46.4. The Group Property of Isometries

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The set $\mathscr{I}$ of isometries of the Gauss plane $\mathbb{C}$ form a group, with the subset $\mathscr{I}_+$ of direct isometries form a normal subgroup. The set $\mathscr{I}_-$ of indirect isometries form a coset with respect to $\mathscr{I}_+$. Both $\mathscr{I}_+$ and $\mathscr{I}$ are non-Abelian groups.

Proof. Let $I_1 : z' = az + b$, $I_2 : z' = cz + d$, $I_3 : z' = a'\overline{z} + b'$, and $I_4 : z' = c'\overline{z} + d'$ where $|a| = |c| = |a'| = |c'| = 1$. Then

$$I_1 \circ I_2 : z' = a(cz + d) + b = (ac)z + (ad + b)$$
$$I_1 \circ I_3 : z' = a(a'\overline{z} + b') + b = (aa')\overline{z} + (ab' + b)$$
$$I_3 \circ I_1 : z' = a'\overline{(az + b)} + b' = (a'\overline{a})\overline{z} + (a'\overline{b} + b')$$
$$I_3 \circ I_4 : z' = a'\overline{(c'\overline{z} + d')} + b' = (a'\overline{c'})z + (a'\overline{d'} + b')$$

and since $|ac| = |a||c| = 1$, $|aa'| = |a||a'| = 1$, $|a'\overline{a}| = |a'||\overline{a}| = 1$, and $|a\overline{c'}| = |a'||\overline{c'}| = |a||c'| = 1$ we have each of these compositions in $\mathscr{I}$ (and these are all possible types of compositions of elements of $\mathscr{I}$), ...

Theorem 46.4 (continued 1)

Proof. ...and so composition really is a binary operation on $\mathscr{I}$. As observed above, function composition is associative, so The Associative Law holds. The identity is $z' = z$ and The Identity Law holds. The inverse of $I_1 : z' = az + b$ is $I_1^{-1} : z' = a^{-1}z - a^{-1}b$ and the inverse of $I_3 : z' = a'\overline{z} + b'$ is $I_3^{-1} : \overline{(a')^{-1}}\overline{z} - \overline{(a')^{-1}b'}$ and The Inverse Law holds. So $\mathscr{I}$ is a group, as claimed.
Notice that $I_1 \circ I_2 \in \mathscr{I}_+$ and $I_1^{-1} \in \mathscr{I}_+$ so for any $I_1, I_2 \in \mathscr{I}_+$ we must have $I_1 \circ I_2^{-1} \in \mathscr{I}_+$ and so by Theorem 44.4, $\mathscr{I}_+$ is a subgroup of $\mathscr{I}$ and so is a group, as claimed.
We now show $\mathscr{I}_-$ is a left coset of $\mathscr{I}_+$. Let $a'\overline{z} + b' \in \mathscr{I}_-$ where $|a'| = 1$. Then $\overline{a'}z + \overline{b'} \in \mathscr{I}_+$, $I^* : z' = \overline{z} \in \mathscr{I}_-$, and left coset $I^*\mathscr{I}_+$ includes $I^* \circ (\overline{a'}z + \overline{b'}) = \overline{(\overline{a'}z + \overline{b'})} = a'\overline{z} + b'$. Since $a'\overline{z} + b'$ is an arbitrary element of $\mathscr{I}_-$ then $\mathscr{I}_- \subset I^*\mathscr{I}_+$. Since the cosets of $\mathscr{I}_+$ partition $\mathscr{I}$, then $\mathscr{I}_- = \mathscr{I} \setminus \mathscr{I}_+$ is a left coset of $\mathscr{I}_+$ and so $\mathscr{I}_+$ only has two cosets. So by Theorem 45.3, "Subgroups of Index Two," $\mathscr{I}_+$ is a normal subgroup of $\mathscr{I}$, as claimed.

Theorem 46.4 (continued 2)

Theorem 46.4. The Group Property of Isometries.
The set $\mathscr{I}$ of isometries of the Gauss plane $\mathbb{C}$ form a group, with the subset $\mathscr{I}_+$ of direct isometries form a normal subgroup. The set $\mathscr{I}_-$ of indirect isometries form a coset with respect to $\mathscr{I}_+$. Both $\mathscr{I}_+$ and $\mathscr{I}$ are non-Abelian groups.

Proof. To establish the non-Abelian claim, notice that $I_5 : z' = iz$ and $I_6 : z' = z + 1$ are in $\mathscr{I}_+ \subset \mathscr{I}$ but $I_5 \circ I_6 : z' = iz + i$ and $I_6 \circ I_5 : z' = iz + 1$, so $I_5 \circ I_6 \neq I_6 \circ I_5$ and $\mathscr{I}_+$, and hence $\mathscr{I}$, are non Abelian, as claimed. □

Corollary 46.4

Corollary 46.4. Let ABC be a triangle and $A'B'C'$ a triangle where $I(A) = A'$, $I(B) = B'$, and $I(C) = C'$ for some direct isometry $I \in \mathscr{I}_+$. If $I_1 \in \mathscr{I}$ is any isometry of the Gauss plane $\mathbb{C}$ then the triangles with vertices $I_1(A), I_1(B), T_1(C)$ and vertices $I_1(A'), I_1(B'), I_1(C')$ are also related by a direct isometry; that is, there is $J \in \mathscr{I}_+$ such that $J(I_1(A)) = I_1(A')$, $J(I_1(B)) = I_1(B')$, and $J(I_1(C)) = I_1(C')$.

Proof. Since $\mathscr{I}$ is a group by Theorem 46.4 then there is $I_1^{-1} \in \mathscr{I}$. Let $J = I_1 \circ I \circ I_1^{-1}$. Then $J(I_1(A)) = I_1 \circ I \circ I_1^{-1}(I_1(A)) = I_1 \circ I(A) = I_1(A')$, $J(I_1(B)) = I_1 \circ I \circ I_1^{-1}(I_1(B)) = I_1 \circ I(A) = I_1(B')$, and $J(I_1(C)) = I_1 \circ I \circ I_1^{-1}(I_1(C)) = I_1 \circ I(C) = I_1(C')$. So J has the desired mapping property and, since $\mathscr{I}_+$ is a normal subgroup of $\mathscr{I}$ by Theorem 46.4, by Theorem 45.2 (with g of Theorem 45.2 equal to I_1^{-1} here) $J \in \mathscr{I}_+$ is the desired direct isometry. □