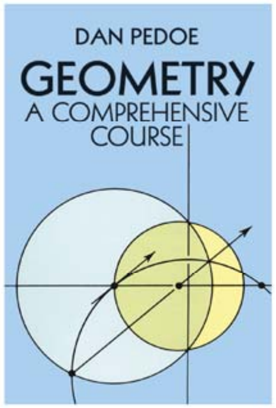
Real Analysis

Chapter V. Mappings of the Euclidean Plane

47. Similarity Transformations and Results—Proofs of Theorems



Theorem 47.3

Theorem 47.3. An Auxiliary Theorem.
Given two pairs of points z_0, z_1 and w_0, w_1 where $|z_0 - z_1| = k|w_0 - w_1| \neq 0$, there is just one mapping of type $\mathscr{S}_+$ and one of type $\mathscr{S}_-$ which maps z_0 to w_0 and maps z_1 to w_1.

Proof. Let $az + b \in \mathscr{S}_+$ with $w_0 = az_0 + b$ and $w_1 = az_1 + b$. Then $w_0 - w_1 = (az_0 + b) - (az_1 + b) = a(z_0 - z_1)$ and

$$a = (w_0 - w_1)/(z_0 - z_1)$$

(this is where we use the facts that $z_0 - z_1 \neq 0$), so that a is uniquely determined in terms of the given w_0, w_1, z_0, z_1. Then

$$b = w_0 - az_0 = w_0 - z_0 \frac{w_0 - w_1}{z_0 - z_1}$$

and b is uniquely determined (also, ...

Theorem 47.3 (continued)

Proof (continued).

$$b = w_1 - az_1 = w_1 - z_1 \frac{w_0 - w_1}{z_0 - z_1} = \frac{w_1(z_0 - z_1) - z_1(w_0 - w_1)}{z_0 - z_1}$$

$$= \frac{w_1 z_0 - z_1 w_0}{z_0 - z_1} = \frac{z_0 w_0 - z_0 w_0 + w_1 z_0 - z_1 w_0}{z_0 - z_1}$$

$$= \frac{w_0(z_0 - z_1) - z_0(w_0 - w_1)}{z_0 - z_1} = w_0 - z_0 \frac{w_0 - w_1}{z_0 - z_1},$$

as expected).

Similarly, for $c\bar{z} + d \in \mathscr{S}_-$ with $w_0 = c\bar{z}_0 + d$ and $w_1 = c\bar{z}_1 + d$. Then $w_0 - w_1 = (c\bar{z}_0 + d) - (c\bar{z}_1 + d) = c(\bar{z}_0 - \bar{z}_1)$ and $c = (w_0 - w_1)/(\bar{z}_0 - \bar{z}_1)$ so that c is uniquely determined in terms of the given w_0, w_1, z_0, z_1. Then $d = w_0 - c\bar{z}_0 = w_0 - \bar{z}_0(w_0 - w_1)/(\bar{z}_0 - \bar{z}_1)$ and d is uniquely determined. □

Theorem 47.4. Similitudes are Collineations

Theorem 47.4. Similitudes are Collineations.
Every similitude of the Gauss plane $\mathbb{C}$ is a collineation.

Proof. Let $z \mapsto z'$ be a similitude. Let ℓ be any line in the Gauss plane $\mathbb{C}$. Choose three points u, v, w on ℓ with v between u and w on ℓ. Then by Lemma 43.A, $|v - w| + |w - u| = |w - u|$. Since the mapping $z \mapsto z'$ is a similitude the for some $k > 0$ we have $|v' - w'| = k|v - w|$, $|v' - u'| = k|v - u|$, $|w' - u'| = k|w - u|$, and so $k^{-1}|v' - w'| + k^{-1}|w' - u'| = k^{-1}|w' - u'|$ or $|v' - w'| + |w' - u'| = |w' - u'|$. Also by Lemma 43.A, u', v', w' are collinear (say they lie on line ℓ') with v' between u' and w' on the line. That is, the collineation maps line ℓ to line ℓ'. Since ℓ is an arbitrary line in $\mathbb{C}$, then the result follows. □

Theorem 47.6. Determination of a Similitude

Theorem 47.6. Determination of a Similitude.
A similitude of the Gauss plane $\mathbb{C}$ is uniquely determined by the assignment of a map of a triangle which is similar to the given triangle. That is, if z_0, z_1, z_2 are noncollinear points with respective images w_0, w_1, w_2 then for any z in the plane, the image of z is uniquely determined from w_0, w_1, w_2.

Proof. Let z_0, z_1, z_2 be noncollinear points in the Gauss plane $\mathbb{C}$. By Lemma 43.A we have: $|z_0 - z_1| + |z_1 - z_2| < |z_0 - z_2|$, $|z_0 - z_1| + |z_0 - z_2| < |z_1 - z_2|$, and $|z_0 - z_2| + |z_1 - z_2| < |z_0 - z_1|$ since the points are noncollinear and equality in any one of these three would imply linearity of the three points. Now $|z_0 - z_2| = k|w_0 - w_1|$, $|z_1 - z_2| = k|w_1 - w_2|$, and $|z_0 - z_2| = k|w_0 - w_2|$ for some $k > 0$ since we have a similitude. So, substituting into the three inequalities and dividing by k, we have $|w_0 - w_1| + |w_1 - w_2| < |w_0 - w_2|$, $|w_0 - w_1| + |w_0 - w_2| < |w_1 - w_2|$, and $|w_0 - w_2| + |w_1 - w_2| < |w_0 - w_1|$ and the points w_0, w_1, w_2 are not collinear.

Theorem 47.6. Determination of a Similitude (continued)

Proof (continued). Let z be a point $\mathbb{C}$ other than z_0, z_1, z_2. Consider the circles C_i with (respective) centers z_i and radii $|z - z_i|$ for $i = 0, 1, 2$. Then the three circles intersect at point z. Since the centers are not collinear, then by Lemma 43.B z is the only point on the three circles. That is, point z is uniquely determined by the three distances $|z - z_0|$, $|z - z_1|$, and $|z - z_2|$. Now triangle $w_0w_1w_2$ is similar to triangle $z_0z_1z_2$ and similarly there is a *unique* point on the intersection of the three circles C_i' centered at w_i with radii $k|z - z_i|$ for $i = 0, 1, 2$; denote the unique point as w. Since the mapping is a similitude then the image of circle C_i is circle C_i' for $i = 0, 1, 2$ and we must have w as the image of z. Since z is an arbitrary point in $\mathbb{C}$ (distinct from z_0, z_1, z_2) then the similitude on $\mathbb{C}$ is uniquely determined. □

Theorem 47.7

Theorem 47.7. There are precisely two similitudes of the Gauss plane $\mathbb{C}$ which map two given points z_0 and z_1 onto the given points w_0 and w_1 where $|w_0 - w_1| = k|z_0 - z_1| \neq 0$.

Proof. Theorem 47.3 gives two such similitudes, one in $\mathscr{S}_+$ and one in $\mathscr{S}_-$. We now show that these are the only such similitudes. Let z be a point in $\mathbb{C}$ that is not collinear with z_0 and z_1. Consider circle C_0 centered at z_0 with radius $|z - z_0|$ and circle C_1 centered at z_1 with radius $|z - z_1|$. Since z lies on both C_0 and C_1 and z is not collinear with the centers of z_0 and z_1 then these circles intersect at two points. Similarly, circle C_0' centered at w_0 with radius $k^{-1}|z - z_0|$ and circle C_1' centered at w_1 with radius $k^{-1}|z - z_1|$ intersect in two points. Since the mapping is a similitude, then z must be mapped to either one or the other of the two points on circles C_0' and C_1'. So there are at most two such similitudes, and the result follows. □

Theorem 47.2. The Main Theorem on Similitudes of the Gauss Plane

Theorem 47.2. The Main Theorem on Similitudes of the Gauss Plane.
The set $\mathscr{S}$ of all similitudes of the Gauss plane $\mathbb{C}$ is composed of two classes, $\mathscr{S}_+$ and $\mathscr{S}_-$. Th class $\mathscr{S}_+$ consists of all similitudes of the form $z' = az + b$ and the class $\mathscr{S}_-$ of all similitudes of the form $z' = c\bar{z} + d$.

Proof. Consider a given similitude of the Gauss plane $\mathbb{C}$. Let z_0 and z_1 be any distinct points in $\mathbb{C}$ with images w_0 and w_1, respectively, under the similitude. By Theorem 47.3, there are two possibilities for the similitude, one in $\mathscr{S}_+$ and one in $\mathscr{S}_-$. By Theorem 47.7, there are only two possibilities for the similitude. So the similitude must be in either $\mathscr{S}_+$ or $\mathscr{S}_-$ and hence every isometry of $\mathbb{C}$ is contained in either $\mathscr{S}_+$ or $\mathscr{S}_-$, as claimed. □

Theorem 47.8. Group Properties of Similitudes

Theorem 47.8. Group Properties of Similitudes.
The similitudes form a group $\mathscr{S}$, the direct similitudes forming a normal subgroup $\mathscr{S}_+$. The opposite similitudes form a coset $\mathscr{S}_-$ with respect to $\mathscr{S}_+$. Neither $\mathscr{S}$ nor $\mathscr{S}_+$ is Abelian. The group of isometries $\mathscr{I}$ is a normal subgroup of $\mathscr{S}$, and $\mathscr{I}_+$ is a normal subgroup of $\mathscr{S}_+$.

Proof. Let $S_1 : z' = az + b$, $S_2 : z' = cz + d$, $S_3 : z' = a'\overline{z} + b'$, and $S_4 : z' = c'\overline{z} + d'$ where a, c, a', c' are nonzero. Then

$$S_1 \circ S_2 : z' = a(cz + d) + b = (ac)z + (ad + b)$$

$$S_1 \circ S_3 : z' = a(a'\overline{z} + b') + b = (aa')\overline{z} + (ab' + b)$$

$$S_3 \circ S_1 : z' = a'\overline{(az + b)} + b' = (a'\overline{a})\overline{z} + (a'\overline{b} + b')$$

$$S_3 \circ S_4 : z' = a'\overline{(c'\overline{z} + d')} + b' = (a'\overline{c'})z + (a'\overline{d'} + b')$$

and we have each of these compositions in $\mathscr{S}$ (and these are all possible types of compositions of elements of $\mathscr{S}$), and so composition really is a binary operation on $\mathscr{S}$.

Theorem 47.8 (continued 1)

Proof (cont.). As observed above, function composition is associative, so The Associative Law holds. The identity is $z' = z$ and The Identity Law holds. The inverse of $S_1 : z' = az + b$ is $S_1^{-1} : z' = a^{-1}z - a^{-1}b$ and the inverse of $S_3 : z' = a'\overline{z} + b'$ is $S_3^{-1} : \overline{(a')^{-1}}\overline{z} - \overline{(a')^{-1}b'}$ and The Inverse Law holds. So $\mathscr{S}$ is a group, as claimed.

Notice that $S_1 \circ S_2 \in \mathscr{S}_+$ and $S_1^{-1} \in \mathscr{S}_+$ so for any $S_1, S_2 \in \mathscr{S}_+$ we must have $S_1 \circ S_2^{-1} \in \mathscr{S}_+$ and so by Theorem 44.4, $\mathscr{S}_+$ is a subgroup of $\mathscr{S}$ and so is a group, as claimed.

We now show $\mathscr{S}_-$ is a left coset of $\mathscr{S}_+$. Let $a'\overline{z} + b' \in \mathscr{S}_-$ where $a \neq 0$. Then $\overline{a'}z + \overline{b'} \in \mathscr{S}_+$, $S^* : z' = \overline{z} \in \mathscr{S}_-$, and left coset $S^*\mathscr{S}_+$ includes $S^* \circ (\overline{a'}z + \overline{b'}) = \overline{(\overline{a'}z + \overline{b'})} = a'\overline{z} + b'$. Since $a'\overline{z} + b'$ is an arbitrary element of $\mathscr{S}_-$ then $\mathscr{S}_- \subset S^*\mathscr{S}_+$. Since the cosets of $\mathscr{S}_+$ partition $\mathscr{S}$, then $\mathscr{S}_- = \mathscr{S} \setminus \mathscr{S}_+$ is a left coset of $\mathscr{S}_+$ and so $\mathscr{S}_+$ only has two cosets. So by Theorem 45.3, "Subgroups of Index Two," $\mathscr{S}_+$ is a normal subgroup of $\mathscr{S}$, as claimed.

Theorem 47.8 (continued 2)

Proof (continued). To establish the non-Abelian claim, notice that $S_5 : z' = iz$ and $S_6 : z' = z + 1$ are in $\mathscr{S}_+ \subset \mathscr{S}$ but $S_5 \circ S_6 : z' = iz + i$ and $S_6 \circ S_5 : z' = iz + 1$, so $S_5 \circ S_6 \neq S_6 \circ S_5$ and $\mathscr{S}_+$, and hence $\mathscr{S}$, are non Abelian, as claimed.

We now show that $\mathscr{I}_+$ is a normal subgroup of $\mathscr{S}_+$. Let $I : a' = az + b \in \mathscr{I}_+$ where $|a| = 1$ and let $S : z' = cz + d \in \mathscr{S}_+$ where $c \neq 0$. Then $S^{-1} : z' = c^{-1}z - c^{-1}d$ and

$$S^{-1} \circ I \circ S = S^{-1} \circ I(cz + d) = s^{-1}(a(cz + d) + b) = S^{-1}((ac)z + (ad + b))$$

$$= c^{-1}((ac)z + (ad + b)) - c^{-1}d = az + c^{-1}(ad + b) - c^{-1}d \in \mathscr{I}_+$$

since $|a| = 1$. Since I is an arbitrary element of $\mathscr{I}_+$ and S is an arbitrary element of $\mathscr{S}_+$, then by Theorem 45.2 $\mathscr{I}_+$ is a normal subgroup of $\mathscr{S}_+$, as claimed.

Theorem 47.8 (continued 3)

Proof (continued). We now show that $\mathscr{I}$ is a normal subgroup of $\mathscr{S}$. With $I_1 : z' = az + b$, $I_3 : z' = a'\overline{z} + b' \in \mathscr{I}$ where $|a| = |a'| = 1$ and $S_1 : z' = cz + d$, $S_3 : z' = c'\overline{z} + d \in \mathscr{S}$ where $c \neq 0 \neq c'$, we have $S_1^{-1} : z' = c^{-1}z - c^{-1}d$ and $S_3^{-1} : z' = \overline{(c')^{-1}}\overline{z} - \overline{(c')^{-1}}\,\overline{d'}$. We know $S_1^{-1} \circ I_1 \circ S_1 \in \mathscr{I}_+ \subset \mathscr{I}$ from above. We also have

$$S_1^{-1} \circ I_3 \circ S_1 = S_1^{-1} \circ I_3(cz + d) = S_1^{-1}(a'\overline{(cz + d)} + b')$$

$$= S_1^{-1}((a'\overline{c})\overline{z} + c^{-1}(a'\overline{d} + b') - c^{-1}d \in \mathscr{I}_- \subset \mathscr{I}$$

since $|c^{-1}a'\overline{c}| = |c|^{-1}|a'||\overline{c}| = |a'| = 1$,

$$S_3^{-1} \circ I_1 \circ S_3 = S_3^{-1} \circ I_1(c'\overline{z} + d') = S_3(a(c'\overline{z} + d') + b')$$

$$= S_3^{-1}((ac')\overline{z} + (ad' + b)) = \overline{(c')^{-1}((ac')\overline{z} + (ad' + b))} - \overline{(c')^{-1}}\,\overline{d'}$$

$$= \overline{(c')^{-1}ac'}z + \overline{(c')^{-1}(ad' + b)} - \overline{(c')^{-1}}\,\overline{d'} \in \mathscr{I}_+ \subset \mathscr{I}$$

since $|\overline{(c')^{-1}ac'}| = |c'|^{-1}|a||c'| = |a| = 1$,

Theorem 47.8 (continued 4)

Proof (continued).

$$S_3^{-1} \circ I_3 \circ S_3 = S_3^{-1} \circ I_3(c'\overline{z} + d') = S_3^{-1}(a'\overline{(c'\overline{z} + d')} + b')$$

$$= S_3^{-1}((a'\overline{c'}z + (a'\overline{d'} + b')) - \overline{(c')^{-1}}\,\overline{b'}$$

$$= (\overline{(c')^{-1}}\,\overline{a'}c')\overline{z} + \overline{(c')^{-1}}(\overline{a'}d' + \overline{b'}) - \overline{(c')^{-1}}\,\overline{b'} \in \mathscr{I}_- \subset \mathscr{I}$$

since $|\overline{(c')^{-1}}\,\overline{a'}c'| = |c|^{-1}|a'||c| = |a| = 1$. Since this covers all possible type of elements of $\mathscr{S}$ and $\mathscr{I}$, we have $S^{-1} \circ I \circ S \in \mathscr{I}$ for all $S \in \mathscr{S}$ and $I \in \mathscr{I}$ and so by Theorem 45.2, $\mathscr{I}$ is a normal subgroup of $\mathscr{S}$, as claimed. □

Theorem 47.9. Condition for Direct Similarity of Triangles

Theorem 47.9. Condition for Direct Similarity of Triangles.
The vertices $z_1, z_2, z_3 \in \mathbb{C}$ of a triangle are mapped by a direct similitude onto the corresponding vertices $w_1, w_2, w_3 \in \mathbb{C}$ of another triangle if and only if

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{w_2 - w_1}{w_3 - w_1}.$$

Proof. First, suppose there is a direct similitude $S : z' = az + b$. Then

$$\frac{w_2 - w_1}{w_3 - w_1} = \frac{S(z_2) - S(z_1)}{S(z_3) - S(z_1)} = \frac{(az_2 + b) - (az_1 + b)}{(az_3 + b) - (az_1 + b)}$$

$$= \frac{az_2 - az_1}{az_3 - az_1} = \frac{z_2 - z_1}{z_3 - z_1} = \frac{z_2 - z_1}{z_3 - z_1},$$

as claimed.

Theorem 47.9 (continued 1)

Proof (continued). Second, suppose $\dfrac{z_2 - z_1}{z_3 - z_1} = \dfrac{w_2 - w_1}{w_3 - w_1}$. Then consider the similitude

$$z' = \frac{w_3 = w_1}{z_3 - z_1}z - \frac{w_3 - w_1}{z_3 - z_1}z_1 + w_1.$$

(We consider "proper triangles" with distinct vertices.) Then

$$\begin{aligned} z_1' &= \frac{w_3 - w_1}{z_3 - z_1}z_1 - \frac{w_3 - w_1}{z_3 - z_1}z_1 + w_1 = w_1, \\ z_2' &= \frac{w_3 - w_1}{z_3 - z_1}z_2 - \frac{w_3 - w_1}{z_3 - z_1}z_1 + w_1 = \frac{w_3 - w_1}{z_3 - z_1}(z_2 - z_1) + w_1 \\ &= (w_3 - w_1)\frac{w_2 - w_1}{w_3 - w_1} + w_1 \text{ by hypothesis} \\ &= w_2 - w_1 + w_1 = w_2, \end{aligned}$$

Theorem 47.9 (continued 2)

Theorem 47.9. Condition for Direct Similarity of Triangles.
The vertices $z_1, z_2, z_3 \in \mathbb{C}$ of a triangle are mapped by a direct similitude onto the corresponding vertices $w_1, w_2, w_3 \in \mathbb{C}$ of another triangle if and only if

$$\frac{z_2 - z_1}{z_3 - z_1} = \frac{w_2 - w_1}{w_3 - w_1}.$$

Proof (continued).

$$\begin{aligned} z_3' &= \frac{w_3 - w_1}{z_3 - z_1}z_3 - \frac{w_3 - w_1}{z_3 - z_1}z_1 + w_1 \\ &= \frac{w_3 - w_1}{z_3 - z_1}(z_3 - z_1) + w_1 = w_3 - w_1 + w_1 = w_3. \end{aligned}$$

So the direct similitude maps z_1, z_2, z_3 to w_1, w_2, w_3, respectively, as claimed. □

Theorem 47.10

Theorem 47.10. Suppose the points z_1, z_2, z_3 are related to the points w_1, w_2, w_3 by a direct similitude, say $w_i = S_1(z_i)$ for $i = 1, 2, 3$. If S is any direct or indirect similitude, then the triangles with vertices $S(z_1), S(z_2), S(z_3)$ and $S(w_1), S(w_2), S(w_3)$ are also related by a direct similitude.

Proof. Since $S_1 \in \mathscr{S}_+$ and $\mathscr{S}_+$ is a normal subgroup of $\mathscr{S}$ by Theorem 47.8, then by Theorem 45.2 $S \circ S_1 \circ S^{-1} \in \mathscr{S}_+$ (that is, $S \circ S_1 \circ S^{-1}$ is a direct similitude) and

$$S \circ S_1 \circ S^{-1}(S(z_i)) = S \circ S_1(z_i) = S(w_i) \text{ for } i = 1, 2, 3.$$

So $S \circ S_1 \circ S^{-1}$ is a direct similitude from the triangle with vertices $S(z_1), S(z_2), S(z_3)$ to $S(w_1), S(w_2), S(w_3)$, as claimed. □