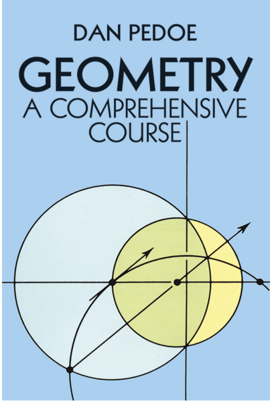
Real Analysis

Chapter V. Mappings of the Euclidean Plane

48. Groups of Translations and Rotations—Proofs of Theorems



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Proof. First, for $T_1 : z' = z + z_1$ and $T_2 : z' = z - z_2$, we have $T_2^{-1} : z' = z - z_2$ and $T_1 \circ T_2^{-1} : z' = z - z_2 + z_1$ is a translation and so by Theorem 44.2, the set of translations is a subgroup of the direct isometries $\mathscr{I}_+$.

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To establish the normal subgroup claim, let $I_1 : z' = az + b \in \mathscr{I}_+$ where $|a| = 1$ and let $T_d : z' = z + d \in \mathscr{T}$. Then $I_1^{-1} : z' = a^{-1}z + a^{-1}b$ and

$$I_1^{-1} \circ T_d \circ I_1 = I_1^{-1} \circ T_d(az + b) = I_1^{-1}((az + b) + d)$$
$$= a^{-1}((az + b) + d) - a^{-1}b = z + a^{-1}d \in \mathscr{T}.$$

Since I_1 is an arbitrary element of $\mathscr{I}_+$ and T_d is an arbitrary element of $\mathscr{T}$ then by Theorem 45.2, $\mathscr{T}$ is a normal subgroup of $\mathscr{I}_+$. □

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Theorem 48.3. The Fixed Point of a Direct Isometry

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Let I_1 be the direct isometry $z' = az + b$ where $|a| = 1$ and $a \neq 1$. Then I_1 has only one fixed point, given by $w = b(1 - a)^{-1}$, and I_1 can be written in the form $z' = a(z - w) + w$, which shows that I_1 is a rotation about w.

Proof. If w is a fixed point under I, then $w = aw + b$ so that $w = b(a - 1)^{-1}$, as claimed. Since $z' = az + b$ then

$$z' = az + (1 - a)w = az + w - aw = a(z - w) + w,$$

as claimed. So I_1 is composed of (1) a translation or w to 0, (2) a rotation about 0 through an angle $\arg(a)$, and (3) a translation of 0 back to w. This accomplishes a rotation about w through an angle $\arg(a)$. □

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The set of rotations about the point w form a group $\mathscr{R}_w$, and the groups $\mathscr{R}_w$, for all values of w, are isomorphic.

Proof. Let $I_1, I_2 \in \mathscr{R}_w$. Say the canonical forms are $I_1 = T_w \circ R \circ T_w^{-1}$ and $I - 2 = T_w \circ R' \circ T_w^{-1}$ where R and R' are rotations about the origin 0. We have $I_1^{-1} = T_w \circ R^{-1} \circ T_w^{-1}$. Now the rotations about the origin form a group by Theorem 46.3, so $R' \circ R^{-1}$ is an element of this group and hence is a rotation about the origin.

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$$I_2 \circ I_1^{-1} = (T_w \circ R' \circ T_w^{-1}) \circ (T_w \circ R^{-1} \circ T_w^{-1}) = T_w \circ (R' \circ R^{-1}) \circ T_w^{-1}$$

is a rotation about w and hence $I_2 \circ I_1^{-1} \in \mathscr{R}_w$. Since I_1 and I_2 are arbitrary elements of $\mathscr{R}_w$ then by Theorem 44.2, $\mathscr{R}_w$ is a subgroup of, say, $\mathscr{I}_+$. So $\mathscr{R}_w$ is a group, as claimed.

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Theorem 48.5 (continued)

Proof (continued). Now consider groups $\mathscr{R}_w$ and $\mathscr{R}_0$ (rotation about the origin). Define $\beta : \mathscr{R}_0 \to \mathscr{R}_w$ as $\beta(R) = T_w \circ R \circ T_w^{-1}$. Since
$\beta : \mathscr{R}_0 \to \mathscr{R}_w$ as $\beta(R) = T_w \circ R \circ T_w^{-1}$. Since R can range over all rotations about 0, the $\beta(R)$ ranges over all rotations about w. That is, β is onto. Clearly, β is one to one. Now for $R, R' \in \mathscr{R}_0$ we have

$$\beta(R \circ R') = T_w \circ (R \circ R') \circ T_w^{-1} = (T_w \circ R \circ T_w^{-1}) \circ (T_w \circ R' \circ T_w^{-1}) = \beta(R) \circ \beta(R')$$

and so β is a group isomorphism. So $\mathscr{R}_w \cong \mathscr{R}_0$. Since this holds for any point w, for $w_1, w_2 \in \mathbb{C}$ we have $\mathscr{R}_{w_1} \cong \mathscr{R}_0$ and $\mathscr{R}_{w_2} \cong \mathscr{R}_0$ and so $\mathscr{R}_{w_1} \cong \mathscr{R}_{w_2}$ (group isomorphisms is an equivalence relation by Exercise 45.1). □

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$$\{\mathscr{R}_w \mid w \in \mathbb{C}\} = \{I \circ \mathscr{R}_0 \circ I^{-1} \mid I \in \mathscr{I}\}.$$

Proof. Every element of $\mathscr{R}_w$ has a canonical form $T_w \circ R \circ T_w^{-1}$ for some $R \in \mathscr{R}_0$ (and conversely $T_w \circ R \circ T_w^{-1}$ is a rotation about w for any $R \in \mathscr{R}_0$) so $\mathscr{R}_w = T_w \mathscr{R}_0 T_w^{-1}$ and all $\mathscr{R}_w$ are conjugates of $\mathscr{R}_0$.

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We now need to show that $I_1 \mathscr{R}_0 I_1^{-1}$ and $I_2 \mathscr{R}_0 I_2^{-1}$ are rotation groups $\mathscr{R}_w$ for some w, where $I_1 \in \mathscr{I}_+$ and $I_2 \in \mathscr{I}_2$. Let $I_1 : z' = az + b$ where $|a| = 1$ be a direct isometry and let $R : z' kz$ where $|k| = 1$ be a rotation about the origin. We have

$$I_1 \circ R \circ I_1^{-1}(z) = I_1 \circ R(a^{-1}z - a^{-1}b)$$

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Proof. ... whereas

$$T_b \circ R \circ T_b^{-1}(z) = T_b \circ R(z-b) = T_b(k(z-b)) = k(z-b)+b = kz-kb+b$$

and so $I_1 \circ R \circ I_1^{-1} = T_b \circ R \circ T_b^{-1} \in \mathscr{R}_b$. So $I_1 \mathscr{R}_0 I_1^{-1} = \mathscr{R}_b$.

Let $I_2 : z' = c\overline{z} + d$ where $|c| = 1$ be an indirect isometry and let $R : z' = kz$, where $|k| = 1$, be a rotation about the origin. We have

$$I_2 \circ R \circ I_2^{-1}(z) = I_2 \circ R(\overline{c^{-1}}\,\overline{z} - \overline{c^{-1}}\,\overline{d}) = I_2(k(\overline{c^{-1}}\,\overline{z} - \overline{c^{-1}}\,\overline{d}))$$

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