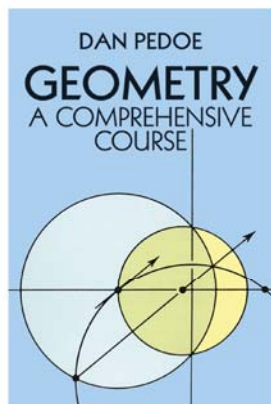


Real Analysis

Chapter V. Mappings of the Euclidean Plane

49. Direct and Indirect Isometries—Proofs of Theorems



Theorem 49.2. Indirect Isometries as Reflections

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An indirect isometry $z' = a\bar{z} + b$ has invariant points if and only if $a\bar{b} + b = 0$. If it has an invariant point, it is a line reflection and has a hole like of invariant points. Every line reflection is an indirect isometry.

Proof. The first claim is shown above.

Suppose indirect isometry $z' = a\bar{z} + b$, where $|a| = 1$, has invariant point w . Then $w = a\bar{w} + b$ and with $z' = a\bar{z} + b$ we have (subtracting)

$$z' - w = (z\bar{z} + b) - (a\bar{w} + b) = a(\bar{z} - \bar{w}) = a\overline{(z - w)}.$$

If $T_w : z' = z + w$ is a translation, and $M_a : z' = a\bar{z}$ is a reflection about the line through the origin which makes an angle $\arg(a)/2$ with the real axis (see Definition 41.3), then

$$\begin{aligned} T_w \circ M_a \circ T_w^{-1}(z) &= T_w \circ M_a(z - w) = T_z(a\overline{(z - w)}) \\ &= a\overline{(z - w)} + w = (z' - w) + w = z'. \end{aligned}$$

Theorem 49.2 (continued 1)

Proof (continued). So the given isometry is $T_w \circ M_a \circ T_w^{-1}$ (a translation of w to 0, a reflection about the line through the origin which makes an angle $\arg(a)/2$ with the real axis, and a translation of 0 to w). This is a reflection about the line through w that makes an angle of $\arg(a)/2$ with the real axis. Hence if the direct isometry has an invariant point then it is a reflection about a line and this line of reflection is invariant, as claimed.

Now suppose we have a line reflection M about a line m . (See the next slide for a picture.) Let w be a point on m and let ℓ be a line through the origin which is parallel to m . Notice that the translation $T_w^{-1} : z' = z - w$ maps line m to line ℓ . Let $M_a : z' = a\bar{z}$ where $|a| = 1$ and $\arg(a)/2$ equals the angle that line ℓ makes with the real axis (so M_a is a reflection about line ℓ). Then $M = T_w \circ M_a \circ T_w^{-1}$ and $M(z) = T_w \circ M_a \circ T_w^{-1}(z) - a\bar{z} - a\bar{w} + w$ where $|a| = 1$ (as shown above) so that M is an indirect isometry. So every line reflection is an indirect isometry, as claimed. \square

Theorem 49.2 (continued 2)

Proof (continued).

