Real Analysis

Chapter V. Mappings of the Euclidean Plane

49. Direct and Indirect Isometries—Proofs of Theorems

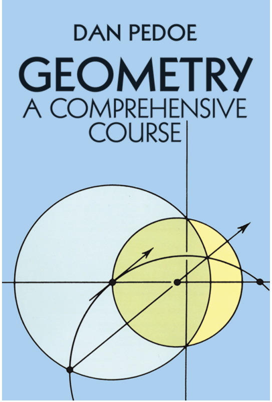

Table of contents

Theorem 49.2. Indirect Isometries as Reflections

Theorem 49.2. Indirect Isometries as Reflections.
An indirect isometry $z' = a\overline{z} + b$ has invariant points if and only if $a\overline{b} + b = 0$. If it has an invariant point, it is a line reflection and has a hole like of invariant points. Every line reflection is an indirect isometry.

Proof. The first claim is shown above.

Theorem 49.2. Indirect Isometries as Reflections

Theorem 49.2. Indirect Isometries as Reflections.
An indirect isometry $z' = a\overline{z} + b$ has invariant points if and only if $a\overline{b} + b = 0$. If it has an invariant point, it is a line reflection and has a hole like of invariant points. Every line reflection is an indirect isometry.

Proof. The first claim is shown above.

Suppose indirect isometry $z' = a\overline{z} + b$, where $|a| = 1$, has invariant point w. Then $w = a\overline{w} + b$ and with $z' = a\overline{z} + b$ we have (subtracting)

$$z' - w = (z\overline{z} + b) - (a\overline{w} + b) = a(\overline{z} - \overline{w}) = a\overline{(z - w)}.$$

Theorem 49.2. Indirect Isometries as Reflections

Theorem 49.2. Indirect Isometries as Reflections.
An indirect isometry $z' = a\overline{z} + b$ has invariant points if and only if $a\overline{b} + b = 0$. If it has an invariant point, it is a line reflection and has a hole like of invariant points. Every line reflection is an indirect isometry.

Proof. The first claim is shown above.

Suppose indirect isometry $z' = a\overline{z} + b$, where $|a| = 1$, has invariant point w. Then $w = a\overline{w} + b$ and with $z' = a\overline{z} + b$ we have (subtracting)

$$z' - w = (z\overline{z} + b) - (a\overline{w} + b) = a(\overline{z} - \overline{w}) = a\overline{(z - w)}.$$

If $T_w : z' = z + w$ is a translation, and $M_a : z' = a\overline{z}$ is a reflection about the line through the origin which makes an angle $\arg(a)/2$ with the real axis (see Definition 41.3), then

$$T_w \circ M_a \circ T_w^{-1}(z) = T_w \circ M_a(z - w) = T_z(a\overline{(z - w)})$$
$$= a\overline{(z - w)} + w = (z' - w) + w = z'.$$

Theorem 49.2. Indirect Isometries as Reflections

Theorem 49.2. Indirect Isometries as Reflections.
An indirect isometry $z' = a\overline{z} + b$ has invariant points if and only if $a\overline{b} + b = 0$. If it has an invariant point, it is a line reflection and has a hole like of invariant points. Every line reflection is an indirect isometry.

Proof. The first claim is shown above.

Suppose indirect isometry $z' = a\overline{z} + b$, where $|a| = 1$, has invariant point w. Then $w = a\overline{w} + b$ and with $z' = a\overline{z} + b$ we have (subtracting)

$$z' - w = (z\overline{z} + b) - (a\overline{w} + b) = a(\overline{z} - \overline{w}) = a\overline{(z - w)}.$$

If $T_w : z' = z + w$ is a translation, and $M_a : z' = a\overline{z}$ is a reflection about the line through the origin which makes an angle $\arg(a)/2$ with the real axis (see Definition 41.3), then

$$T_w \circ M_a \circ T_w^{-1}(z) = T_w \circ M_a(z - w) = T_z(a\overline{(z - w)})$$
$$= a\overline{(z - w)} + w = (z' - w) + w = z'.$$

Theorem 49.2 (continued 1)

Proof (continued). So the given isometry is $T_w \circ M_a \circ T_w^{-1}$ (a translation of w to 0, a reflection about the line through the origin which makes an angle $\arg(a)/2$ with the real axis, and a translation of 0 to w). This is a reflection about the line through w that makes an angle of $\arg(a)/2$ with the real axis. Hence if the direct isometry has an invariant point then it is a reflection about a line and this line of reflection is invariant, as claimed.

Now suppose we have a line reflection M about a line m. (See the next slide for a picture.) Let w be a point on m and let ℓ be a line through the origin which is parallel to m. Notice that the translation $T_w^{-1} : z' = z - w$ maps line m to line ℓ. Let $M_a : z' = a\bar{z}$ where $|a| = 1$ and $\arg(a)/2$ equals the angle that line ℓ makes with the real axis (so M_a is a reflection about line ℓ).

Theorem 49.2 (continued 1)

Proof (continued). So the given isometry is $T_w \circ M_a \circ T_w^{-1}$ (a translation of w to 0, a reflection about the line through the origin which makes an angle $\arg(a)/2$ with the real axis, and a translation of 0 to w). This is a reflection about the line through w that makes an angle of $\arg(a)/2$ with the real axis. Hence if the direct isometry has an invariant point then it is a reflection about a line and this line of reflection is invariant, as claimed.

Now suppose we have a line reflection M about a line m. (See the next slide for a picture.) Let w be a point on m and let ℓ be a line through the origin which is parallel to m. Notice that the translation $T_w^{-1} : z' = z - w$ maps line m to line ℓ. Let $M_a : z' = a\overline{z}$ where $|a| = 1$ and $\arg(a)/2$ equals the angle that line ℓ makes with the real axis (so M_a is a reflection about line ℓ). Then $M = T_w \circ M_a \circ T_w^{-1}$ and $M(z) = T_w \circ M_a \circ T_w^{-1}(z) - a\overline{z} - a\overline{w} + w$ where $|a| = 1$ (as shown above) so that M is an indirect isometry. So every line reflection is an indirect isometry, as claimed. □

Theorem 49.2 (continued 1)

Proof (continued). So the given isometry is $T_w \circ M_a \circ T_w^{-1}$ (a translation of w to 0, a reflection about the line through the origin which makes an angle $\arg(a)/2$ with the real axis, and a translation of 0 to w). This is a reflection about the line through w that makes an angle of $\arg(a)/2$ with the real axis. Hence if the direct isometry has an invariant point then it is a reflection about a line and this line of reflection is invariant, as claimed.

Now suppose we have a line reflection M about a line m. (See the next slide for a picture.) Let w be a point on m and let ℓ be a line through the origin which is parallel to m. Notice that the translation $T_w^{-1} : z' = z - w$ maps line m to line ℓ. Let $M_a : z' = a\overline{z}$ where $|a| = 1$ and $\arg(a)/2$ equals the angle that line ℓ makes with the real axis (so M_a is a reflection about line ℓ). Then $M = T_w \circ M_a \circ T_w^{-1}$ and
$M(z) = T_w \circ M_a \circ T_w^{-1}(z) - a\overline{z} - a\overline{w} + w$ where $|a| = 1$ (as shown above) so that M is an indirect isometry. So every line reflection is an indirect isometry, as claimed. □

Theorem 49.2 (continued 2)

Proof (continued).