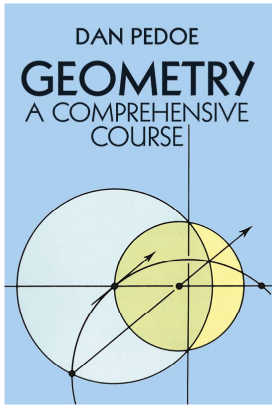
Real Analysis

Chapter V. Mappings of the Euclidean Plane

50. More Isometries and Similitudes—Proofs of Theorems



Theorem 50.1

Theorem 50.1. Every involutory isometry of the Gauss plane $\mathbb{C}$ is either a line reflection, a half-turn, or the identity.

Proof. By Theorem 43.1, "The Main Theorem on Isometries of the Gauss Plane," there are only two types of isometries of the Gauss plane, direct and indirect isometries. For direct isometry $M : z' = az + b$ where $|a| = 1$, the square of the transformation is $z' = a(az + b) + b = a^2 z + ab + b$, so this is involutory and is the identity transformation if only if $a^2 = 1$ and $ab + b = 0$. So we could have $a = 1$ and $b = 0$ which implies M is the identity. We could also have $a = -1$ and b unrestricted, which implies $M : z' = -z + b = -(z - b/2) + b/2$ which is a half-turn (see Section 48). So direct isometry M is involutory, as claimed.

Theorem 50.1 (continued)

Theorem 50.1. Every involutory isometry of the Gauss plane $\mathbb{C}$ is either a line reflection, a half-turn, or the identity.

Proof (continued). For indirect isometry $M : z' = a\bar{z} + b$ where $|a| = 1$, the square of the isometry is

$$z' = a\overline{(a\bar{z} + b)} + b = a\bar{a}z + a\bar{b} + b = |a|^2 z + a\bar{b} + b = z + a\bar{b} + b,$$

so this is the identity if only if $a\bar{b} + b = 0$. By Theorem 49.2, "Indirect Isometries as Reflections," this means that M must be a line reflection, as claimed. □

Theorem 50.2. The Composition of Two Reflections

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The composition of reflections in lines ℓ and m result in (i) a translation if and only if the lines are parallel, or (ii) a rotation about the point of intersection if and only if the lines intersect.

Proof. Let the reflection in ℓ be given by $\bar{z} = a\bar{z} + b$ with $|a| = 1$ and $a\bar{b} + b = 0$ by Theorem 49.2. The line ℓ makes an angle of $\arg(a)/2$ with the real axis. Let the reflection in m be given by $z' = c\bar{z} + d$ with $|c| = 1$, $c\bar{d} + d = 0$, and line m makes an angle of $\arg(c)/2$ with the real axis. The composition of the reflections is

$$z' = c\overline{(a\bar{z} + b)} + d = c\bar{a}z + c\bar{b} + d \in \mathscr{I}_+.$$

Now $|c\bar{a}| = 1$, so either $c\bar{a} = 1$, or $|c\bar{a}| = 1$ and $c\bar{a} \neq 1$.

(i) Suppose $c\bar{a} = 1$. Then the composition is $z' = z + c\bar{b} + d$, a translation. Also, $(c\bar{a})a = 1a$ or $c|a|^2 = a$ or $c = a$. So $\arg(a) = \arg(c)$ and line ℓ and m are parallel.

Theorem 50.2 (continued 1)

Proof (continued). Conversely, if line ℓ and m are parallel then $\arg(a) = \arg(c)$ and, since $|a| = |c|$ then $a = c$ and so $\bar{a}c = \bar{c}c = |c|^2 = 1$ and the composition $z' = z + c\bar{b} + d$ is a translation. So the composition is a translation if and only if the lines are parallel.

(ii) $|c\bar{a}| = 1$ and $c\bar{a} \neq 1$. Then the composition $z' = c\bar{z}z + c\bar{b} + d$ is a direct isometry which is not a translation. Then by Theorem 48.3, "The Fixed Point of a Direct Isometry," the composition is a rotation about the fixed point of the composition. The fixed point satisfies $z = c\bar{a}z + c\bar{b} + d$ or $(z - c\bar{a})z = c\bar{b} + d$ or $z = (c\bar{b} + d)/(1 - c\bar{a})$. In Exercise 50.1, it is to be shown that this point is left fixed by both reflections. Since the only points fixed by a reflection are the points on the line of reflection, Exercise 50.1 implies that the composition is a rotation about the unique point of intersection of lines ℓ and m (since $c\bar{a} \neq 1$ then $c \neq a$ and $\arg(c) = \arg(a)$ because $|a| = |c| = 1$; so that ℓ and m are not parallel).

Theorem 50.2 (continued 2)

Theorem 50.2. The Composition of Two Reflections.
The composition of reflections in lines ℓ and m result in (i) a translation if and only if the lines are parallel, or (ii) a rotation about the point of intersection if and only if the lines intersect.

Proof (continued). Conversely, if ℓ and m are not parallel then $\arg(a) \neq \arg(c)$ and $c\bar{a} \neq 1$. So the composition is a rotation about the point of intersection, as just shown. □

Theorem 50.3. A Reflection Glide is also a Glide Reflection

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A reflection in a line ℓ followed by a translation T_b results in an opposite isometry without invariant points if and only if that ℓ is no perpendicular to the position vector b. An opposite isometry without fixed points is equivalent to a glide reflection, that is to a reflection in a line followed by a translation parallel to the line.

Proof. Let the mapping be $z' = a\bar{z} + b$. If the mapping has fixed points then, by Theorem 49.2, $a\bar{b} + b = 0$. Then $a\bar{b} = -b$ and $\arg(a) + \arg(\bar{b}) = \arg(-b) = \text{arc}(b) \pm \pi$ so that $\arg(a) = 2\arg(b) \pm \pi$ and $\arg(a)/2 = \arg(b) \pm \pi/2$. Since $\arg(a)/2$ is the angle the line reflection ℓ makes with the real axis, so ℓ is perpendicular to the "position vector" b. So if ℓ is not perpendicular to the position vector b then the mapping has no invariant points, as claimed.

Theorem 50.3 (continued 1)

Proof (continued). Conversely, if line ℓ is perpendicular to position vector b then $\arg(a)/2 = \arg(b) \pm \pi/2$ and we have $a\bar{b} + b = 0$. So by Theorem 49.2, the mapping has fixed points. In fact, the mapping has an invariant line. Consider the line ℓ' parallel to line ℓ and a distance $|b|/2$ from ℓ in the "direction" b (so that for any point u on ℓ the point $u + b/2$ is on line ℓ'). Consider arbitrary point $u + b/2$ on ℓ' where u is on ℓ. The reflection about line ℓ maps $u + b/2$ to $u - b/2$ and then the translation T_b maps $u - b/2$ to $u + b/2$. So the mapping has line ℓ' invariant.

Now suppose $z' = a\bar{z} + b$, where $|a| = 1$, is an opposite isometry without fixed points. Then by Theorem 49.2, $a\bar{b} + b \neq 0$. Now $a\bar{z} + b = a\overline{(z - b/2)} + b/2 + (a\bar{b} + b)/2$. Set $d = (a\bar{b} + b)/2$ so that $d \neq 0$. So we have $z' = a\bar{z} + b$ as the composition $T_d \circ T_{b/2} \circ M_a \circ T_{b/2}^{-1}$ since

$$T_d \circ T_{b/2} \circ M_a \circ T_{b/2}^{-1}(z) = T_d \circ T_{b/2} \circ M_a(z - b/2) = T_d \circ T_{b/2}(a\overline{(z - b/2)})$$

$$= T_d(a\overline{(z - b/2)} + b/2) = a\overline{(z - b/2)} + b/2 + d = z'.$$

Theorem 50.3 (continued 2)

Proof (continued). Now $T_{b/2} \circ M_a \circ T_{b/2}^{-1}$ is the canonical form for a line reflection by Theorem 49.2 where the line makes an angle $\arg(a)/2$ with the real axis. So the mapping is a line reflection followed by a translation. We need to show that "position vector" d is parallel to line ℓ. We have, since $|a| = a\overline{a} = 1$,

$$\left(\frac{d}{|d}\right)^2 - \frac{d^2}{d\overline{d}} = \frac{d}{\overline{d}} = \frac{a\overline{b}+b}{\overline{a}b+\overline{b}} = \frac{a(a\overline{b}+b)}{a(\overline{a}b+\overline{b})} = \frac{a(a\overline{b}+b)}{b+a\overline{b}} = a$$

and so $2\arg(d) = \arg(d^2) = \arg(d^2/|d|^2) = \arg(a)$, or $\arg(d) = \arg(a)/2$. So position vector d is parallel to line ℓ and the mapping is a reflection about a line ℓ followed by a translation parallel to line ℓ. That is, the opposite isometry without fixed points is a glide reflection , as claimed. □

Theorem 50.4.I. Hjelmslev's Theorem

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Suppose the points P on a line are mapped by a plane isometry onto the points P' of another line. Then the midpoints of the line segments PP' either coincide or are distinct and collinear.

Proof. First, suppose the isometry is a direct isometry. A direct isometry is either a translation or it is a rotation about a fixed point (see Theorem 42.1). If it is a translation, then it maps a given line to a line parallel to the given line. Let points $A, B, C, \ldots$ lie on a line, and suppose they are mapped onto points $A', B', C', \ldots$, respectively, by the translation. Then the points $A', B', C', \ldots$ are collinear by Theorem 43.3. Also, the midpoints of the segments $AA', BB', CC', \ldots$ lie on a line (see Figure 50.4), as claimed.

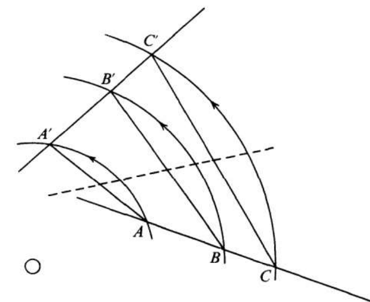

Figure 50.4

Theorem 50.4.I. Hjelmslev's Theorem (continued 1)

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Suppose the points P on a line are mapped by a plane isometry onto the points P' of another line. Then the midpoints of the line segments PP' either coincide or are distinct and collinear.

Proof (continued). If the direct isometry is a rotation through an angle equal to π, then the rotation is a "half-turn" and the midpoints coincide with the center of rotation. We address other rotations below.

If the isometry is indirect, then by Note 50.B it is either a reflection in a line ℓ (in which case the midpoints of $AA', BB', CC' \ldots$ lie on ℓ, or is a glide-reflection (i.e., a reflection in ℓ followed by a translation parallel to ℓ), and again the midpoints of $AA', BB', CC', \ldots$ lie on ℓ.

Theorem 50.4.I. Hjelmslev's Theorem (continued 2)

Proof (continued). We still need to address the case of a direct isometry which is a rotation about a point through an angle other than π. We do so algebraically in such a way as to prove the result for all direct isometries (hence encompassing some of the above work already done). Let the points $A, B, C, \ldots$ be represented by the complex numbers $a, b, c, \ldots$, respectively, and let the points $A', B', C', \ldots$ be represented by the complex numbers $a', b', c', \ldots$, respectively. If c is the complex number which represents any point C on the line AB, then $c = ta + (1-t)b$ where t is real and the ratio of the real numbers $1-t : t$ equals the ratio of the signed lengths of the segments $\overline{AC} : \overline{CB}$ by Theorem 3.1. Let c' be the image of c under the direct isometry $z' = pz + q$ where $|p| = 1$. Then

$$c' = pc + q = p(ta + (1-t)b) + q$$

$$= t(pa+q) + (1-t)(pb+q) = ta' + (1-t)b'. \qquad (*)$$

Theorem 50.4.I. Hjelmslev's Theorem (continued 3)

Theorem 50.4.I. Hjelmslev's Theorem.
Suppose the points P on a line are mapped by a plane isometry onto the points P' of another line. Then the midpoints of the line segments PP' either coincide or are distinct and collinear.

Proof (continued). Next, suppose that real numbers k and k' satisfy $k + k' = 1$. Then by $(*)$ we have

$$k'c + kc' = k'(ta + (1-t)b) + k(ta' + (1-t)b')$$
$$= t(k'a + ka') + (1-t)(kb' + k'b). \qquad (**)$$

Suppose k and k' are now fixed. If $k'a + ka' \neq k'b + kb'$, then the points $k'c + kc'$ "move" on a line as t varies and are distinct for distinct values of t, as claimed.

Theorem 50.4.I. Hjelmslev's Theorem (continued 4)

Proof (continued). If $k'a + ka' = k'b + kb'$ then by $(**)$ we have $k'c + kc' = k'a + ka' = k'b + kb'$. Then all the points which divide the segment CC' in the ratio $k : k'$ coincide. In terms of a, b, a', b' this implies that $k'(a-b) = -k(a'-b')$. So if A and B are any two points on the one line, and A' and B' are the images of A and B on the other line, then we have for these fixed values of k and k' that $k'\overrightarrow{AB} = -k\overrightarrow{A'B'}$. In this case the line containing points A and B is parallel to the line containing points A' and B' (we are using the difference of two complex numbers as a direction vector of a line here). Since $k'(a-b) = -k(a'-b')$ and we are dealing with an isometry (so that $|a-b| = |a'-b'|$), then we must have $k = k' = 1/2$ and since $k'a + ka' = k'b + kb'$ (or $(a+a')/2 = (b+b')/2$) then the midpoints coincide, as claimed.

The computations are similar for an indirect isometry, based on the fact that the conjugate of a real number (such as t, $1-t$, k ad k') is the number itself. □

Theorem 50.5.

Theorem 50.5. A proper direct similitude $z' = az + b$, where $|a| \neq 1$, has a unique fixed point. A proper indirect similitude $z' = c\overline{z} + d$, where $|c| \neq 1$, has a unique fixed point.

Proof. If w is a fixed point of $z' = az + b$, then $w = aw + b$ or $w(1-a) = b$. Since $a \neq 1$, then there is a unique value of w, namely $w = b/(1-a)$.

If w is a fixed point of $z' = c\overline{z} + d$, then $w = c\overline{w} + d$ or, conjugating both sides of this equation, $\overline{w} = \overline{c}w + \overline{d}$. Now replacing $\overline{w}$ with $\overline{c}w + \overline{d}$ in this last equation, we have $w = c(\overline{c}w + \overline{d}) + d$. Then $w(1 - c\overline{c}) = c\overline{d} + d$ or $w(1 - |c|^2) = c\overline{d} + d$. Since $|c| \neq 1$ then there is a unique value of w, namely $w = (c\overline{d} + d)/(1 - |c|^2)$. □