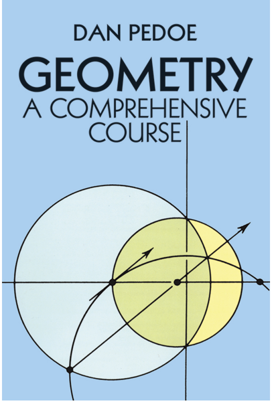
Real Analysis

Chapter V. Mappings of the Euclidean Plane

51. Line Reflections and Isometries as Reflections—Proofs of Theorems



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Theorem 51.1

Theorem 51.1. Every product (i.e., composition) of three line reflections is either a line reflection or a glide reflection.

Proof. Let the three reflections be the mappings $z' = a_i\overline{z} + b_i$ for $i = 1, 2, 3$. The composition of the mappings is:

$$z' = a_3\overline{(a_2\overline{(a_1\overline{z} + b_1)} + b_2)} + b_3 = a_3(\overline{a_2}(a_1\overline{z} + b_1) + \overline{b_2}) + b_3$$

$$= a_1\overline{a_2}a_3\overline{z} + \overline{a_2}a_3b_1 + a_3\overline{b_2} + b_3 = (a_1\overline{a_2}a_3)\overline{z} + (\overline{a_2}a_3b_1 + a_3\overline{b_2} + b_3),$$

and this is an indirect isometry. By Note 50.B, an indirect isometry is either a line reflection (when it has at least one fixed points, by Theorem 49.2) or a glide reflection (when it has no fixed points, by Theorem 50.3), as claimed. □

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Theorem 50.2. Condition for Two Lines to be Perpendicular

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Two distinct lines ℓ and m are perpendicular to each other if and only if the product (i.e., composition) of the reflections in ℓ and m is involutory, and not the identity.

Proof. First, let line ℓ be perpendicular to line m, and let w be the point of intersection. The canonical form of the reflection in ℓ is (as seen in the proof of Theorem 49.2) $T_w^{-1} M_a T_w$, and the reflection in m is $T_w^{-1} M_c T_w$ where T_w is the translation $z' = z + w$, M_a is a reflection $z' = a\overline{z}$, and M_c is a reflection $z' = c\overline{z}$, the line ℓ and m making angles of $\arg(a)/2$ and $\arg(c)/2$, respectively, with the X-axis.

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$$M = (T_w^{-1}M_aT_w)(T_w^{-1}M_cT_w) = T_w^{-1}M_aM_cT_w.$$

Notice that $M^2 = T_w^{-1}(M_aM_c)^2T_w$, so if we show that M_aM_c is involutory, then we have that M is involutory. Now $M_aM_b$is the mapping $z' = z\overline{c}z$.

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Theorem 50.2 (continued 1)

Proof (continued). Since ℓ is perpendicular to m, then $\arg(a)/2 - \arg(c)/2 = \pm\pi/2$, or $\arg(a) - \arg(c) = \pm\pi$ or $\arg(a/c) = \pm\pi$. Since $|a| = |c| = 1$ (see Definition 41.3 of reflection), then $|a/c| = 1$ and $a/c = \cos\pi \pm i\sin w = -1$ so that $a = -c$. Then $\overline{a}c = -\overline{c}c = -|c|^2 = -1$. So $M = T_w^{-1}M_aM_cT_w$ is the transformation

$$z' = (T_wM_cM_aT_w^{-1})(z) = (T_wM_cM_a)(T_w^{-1}(z)) = (T_wM_cM_a)(z-w)$$

$$= (T_wM_c)(M_a(z-w)) = (T_wM_c(a\overline{(z-w)})) = T_w(M_c(a\overline{(z-w)}))$$

$$= T_w(c\overline{a\overline{(z-w)}}) = c\overline{a\overline{(z-w)}} + w$$

$$= c\overline{a}(z-w) + w = \overline{a}cz - \overline{a}cw + w = -z + 2w. \qquad (*)$$

Notice that this is involutory (because $-(-z+2w)+2w = z$) and not the identity, as claimed.

Theorem 50.2 (continued 1)

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Theorem 51.2. Condition for Two Lines to be Perpendicular. Two distinct lines ℓ and m are perpendicular to each other if and only if the product (i.e., composition) of the reflections in ℓ and m is involutory, and not the identity.

Proof (continued). Second for the converse, suppose M is involutory and not the identity. By Theorem 50.2 the distinct lines are not parallel, otherwise M would be a non-zero translation and so not involutory. Let ℓ and m intersect at the point w. Then as seen in $(*)$ above, we have $z' = \overline{a}cz - \overline{a}cw + w$. This is a direct isometry (and so not a line reflection) and, since it is by hypothesis involutory, then by Theorem 50.1 it is either a half-turn (i.e., a point-reflection) or the identity. But M is not the identity by hypothesis, so it must be a half-turn. So $z' - w = \overline{a}cz - \overline{a}cw = \overline{a}c(z - w) = -(z - w)$ and hence $\overline{a}c = -1$. But from $\overline{a}c = -1$ we have $\arg(\overline{a}) + \arg(c) = \pm\pi$ or $\arg(a)/2 - \arg(c)/2 = \pm pi/2$. Therefore ℓ and m are perpendicular, as claimed. $\square$

Theorem 50.2 (continued 2)

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Theorem 51.3. The Composition of Two Reflections

Theorem 51.3. The product (i.e., composition) of the reflections in a point w and in a line ℓ is involutory if and only if w lies on ℓ.

Proof. First, suppose point w lies on line ℓ. Let the reflection in ℓ be $M_\ell : z' = a\overline{z} + b$ where $a\overline{b} + b = 0$ (by Theorem 49.2) and let the reflection in w be $M_w : z' = -z + 2w$ (by Note 48.C). Then the product $M_w M_\ell = M_\ell \circ M_w$ is M, say, were M is given by

$$z' = a\overline{(-z + 2w)} + b = -a\overline{z} + 2a\overline{w} + b. \qquad (*)$$

Since w is on ℓ, then w is invariant under M_ℓ; that is, $w = a\overline{w} + b$.

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$$\begin{aligned} z' &= -a\overline{(-a\overline{z} + 2a\overline{w} + b)} + 2a\overline{w} + b \\ &= |a|^2 z - 2|a|^2 w - a\overline{b} + 2a\overline{w} + b \\ &= z - 2w + 2a\overline{w} + (b) + b \text{ since } |a| = 1 \text{ and } a\overline{b} + b = 0 \\ &= z - 2w + 2(a\overline{w} + b) = z - 2w + 2w \text{ since } w = a\overline{w} + b \\ &= z. \end{aligned}$$

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Proof (continued). Second, suppose that M^2 is involutory. Since M is also an indirect isometry by $(*)$ (so it is not a half-turn or the identity), then by Theorem 50.1 M must be a line reflection and so has fixed points. By Note 49.A, the condition for $z' = p\overline{z} + q$ to have a first point is $p\overline{q} + q = 0$. Since we have M as the mapping $z' = -a\overline{z} + 2a\overline{w} + b$ by $(*)$, then the condition $p\overline{q} + q = 0$ implies for M that $-a\overline{(2a\overline{w} + b)} + 2a\overline{w} + b = 0$ or $-2|a|^2\overline{w} - a\overline{b} + 2a\overline{w} + b = 0$ or (since $|a| = 1$) $-2\overline{w} - a\overline{b} + 2a\overline{w} + b = 0$. This gives $-2w - (a\overline{b} + b) + 2a\overline{w} + 2b = 0$ or, since $a\overline{b} + b = 0$, $-2w + 2a\overline{w} + 2b = 0$, or $w = a\overline{w} + b$.

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Theorem 51.A

Theorem 51.A. A translation can be written as the product (composition) of reflections in parallel lines, and a rotation can be written as the product of reflections in two lines through the center of rotation.

Proof. First, consider a translation. If the translation is the identity (i.e., a translation by an amount of 0), then the translation is the product of two reflections about the same line, and the claim holds. If the translation is of the form $z' = z + b$ where $b \neq 0$, the we can solve for a in the equation $a\overline{b} + b = 0$ to get $a = -b/\overline{b}$ (and so $|a| = 1$).

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Theorem 51.A (continued 1)

Theorem 51.A. A translation can be written as the product (composition) of reflections in parallel lines, and a rotation can be written as the product of reflections in two lines through the center of rotation.

Proof (continued). Also $M_a : z' = a\overline{z}$ is a reflection about a line passing through the origin 0, which makes and angle of $\arg(a)/2$ with the real axis. Composing $z' = a\overline{z} + b$ with $M_a : z' = a\overline{z}$ we have $z' = a(\overline{(a\overline{z})} + b = z + b$ (since $|a| = 1$). Hence the given translation $z' = z + b$ is the product of reflection about parallel lines (as claimed), where the lines are a distance $|b|/2$ apart with one line passing through the origin and both lines making an angle of $\arg(a)/2 = \arg(-b/\overline{b})$ with the real axis.

Theorem 51.A (continued 2)

Theorem 51.A. A translation can be written as the product (composition) of reflections in parallel lines, and a rotation can be written as the product of reflections in two lines through the center of rotation.

Proof (continued). Now consider a rotation. By Theorem 48.3, a rotation is of the form $z' = a(z - w) + w$ where $|a| = 1$ and w is the center of the rotation. So the isometry is of the form $T_w^{-1} R_a T_w$ where w is the center of rotation, and R_a is the rotation $z' = az$ (using Pedoe's notation here; in terms of compositions, the isometry is of the form $T_w \circ R_a \circ T_w^{-1}$; see Note 48.B). Now we have $T_w \circ R_a \circ T_w^{-1} = (T_w \circ M_a \circ T_w^{-1}) \circ (T_w \circ M_1 \circ T_w^{-1})$ where M_1 is the reflection $z' = \overline{z}$ and M_a is the reflection $z' = z\overline{z}$ (so that $R_a = M_a \circ M_1$). Now both $(T_w \circ M_a \circ T_w^{-1})$ and $(T_w \circ M_1 \circ T_w^{-1})$ are reflections about lines through w by Theorem 49.2, "Indirect Isometries as Reflections (see also the definition of canonical form of such an isometry), so the rotations has been written as a product of reflections in two lines through the center of rotation, as claimed. □

Theorem 51.A (continued 2)

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Proof (continued). Now consider a rotation. By Theorem 48.3, a rotation is of the form $z' = a(z - w) + w$ where $|a| = 1$ and w is the center of the rotation. So the isometry is of the form $T_w^{-1} R_a T_w$ where w is the center of rotation, and R_a is the rotation $z' = az$ (using Pedoe's notation here; in terms of compositions, the isometry is of the form $T_w \circ R_a \circ T_w^{-1}$; see Note 48.B). Now we have $T_w \circ R_a \circ T_w^{-1} = (T_w \circ M_a \circ T_w^{-1}) \circ (T_w \circ M_1 \circ T_w^{-1})$ where M_1 is the reflection $z' = \overline{z}$ and M_a is the reflection $z' = z\overline{z}$ (so that $R_a = M_a \circ M_1$). Now both $(T_w \circ M_a \circ T_w^{-1})$ and $(T_w \circ M_1 \circ T_w^{-1})$ are reflections about lines through w by Theorem 49.2, "Indirect Isometries as Reflections (see also the definition of canonical form of such an isometry), so the rotations has been written as a product of reflections in two lines through the center of rotation, as claimed. $\square$

Theorem 51.4. Isometries as Reflections

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Every isometry is the product (i.e., composition) of at most three reflections. If the isometry has a fixed point, at most two line reflections produce the isometry.

Proof. First, suppose that the isometry is direct. Then by Theorem 48.4, "Direct Isometries and Rotations," the isometry is either a translation of a rotation. By Theorem 51.A a translation can be expressed as the product of reflections in two parallel lines which are perpendicular to the vector which give the direction of the translation and so the first claim holds for a translation. Also by Theorem 51.A, a rotation through an angle $2A$ about a point O is the product of reflections about two lines intersection at O at an angle of A. So a direct isometry can be expressed as the product of *two* reflections and the claim holds.

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Every isometry is the product (i.e., composition) of at most three reflections. If the isometry has a fixed point, at most two line reflections produce the isometry.

Proof (continued). Now suppose that the isometry is indirect. Then the isometry is either a line reflection (when it has a fixed point) or a glide reflection (when it has not fixed points); see Note 50.B. If the isometry *is* a line reflection, then the claim holds. If the isometry is a glide reflection then it is a reflection in a line followed by a translation parallel to the line by Theorem 50.3; the translation is a product of reflections by Theorem 51.A (as described above), so that the isometry is then a product of three reflections, and the claim holds. This establishes the first claim.

Theorem 51.4. Isometries as Reflections (continued 2)

Theorem 51.4. Isometries as Reflections.
Every isometry is the product (i.e., composition) of at most three reflections. If the isometry has a fixed point, at most two line reflections produce the isometry.

Proof (continued). If the isometry has a fixed point, then it is either a rotation (when it is a direct isometry, by Note 48.A) or a reflection about a line by (when it is an indirect isometry, by Theorem 49.2). A rotation is the product of two reflections by Theorem 51.A (as described above), so for an isometry with a fixed point at most two line reflections produce the isometry, as claimed. □