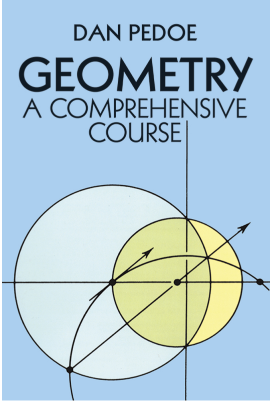
Real Analysis

Chapter VI. Mappings of the Inversive Plane

52. Möbius Transformations—Proofs of Theorems



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Proof. As described above, any line or circle in the Gauss plane is of the form $|z-p| = k|z-q|$ where k is a positive constant and p and q are complex constants. If the Möbius transformation is $Z = \dfrac{az+b}{cz+d}$, then from Note 52.A we have $z = \dfrac{-dZ+b}{cZ-a}$.

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$$\left|\frac{-dZ + b}{cZ - a} - p\right| = k\left|\frac{-dZ + b}{cZ - a} - q\right|.$$

Multiplying both sides of this equation by $|cZ - a|$ gives $|-dZ + b - p(cZ - a)| = k|-dZ + b - q(cZ - a)|$ or $|Z(-d - pc) + b + pa| = k|Z(-d - qc) + b + qa|$.

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Proof (continued). $\ldots |Z(-d-pc)+b+pa| = k|Z(-d-qc)+b+qa|$. Dividing both sides of this equation by $|(-cp-d)(-cq-d)|$ gives

$$\left|Z - \frac{ap+b}{cp+d}\right| \frac{1}{|cq+d|} = k \left|Z - \frac{aq+b}{cq+d}\right| \frac{1}{|cp+d|}$$

or

$$\left|Z - \frac{ap+b}{cp+d}\right| = k \left|\frac{cq+d}{cp+d}\right| \left|Z - \frac{aq+b}{cq+d}\right|.$$

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This if the form $|Z-p'| = k'|Z-q'|$, where p' and q' are the images of the points p and q under the Möbius transformation, and $k' = k|cq+d|/|cp+d|$, and so is also an Apollonius circle, as claimed. □

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Theorem 52.A

Theorem 52.A. Every Möbius transformation is a composition of translations, dilations, rotations, and inversions.

Proof. Let $z' = \dfrac{az+b}{cz+d}$ be a Möbius transformation. Then we have by long division that $z' = \dfrac{az+b}{cz+d} = \dfrac{a}{c} + \dfrac{bc-ad}{c(cz+d)}$, and so $z' = \dfrac{a}{c} + \dfrac{bc-ad}{c^2}\left(\dfrac{1}{z+d/c}\right)$.

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$$z' = \frac{az+b}{cz+d} = (T_{a/c} \circ D \circ N \circ T_{d/c})(z) = T_{a/c}(D(N(T_{d/c}(z)))),$$

as claimed. □

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Theorem 52.5

Theorem 52.5. Möbius transformations form a group $\mathscr{B}$ under composition of mappings. If B and C are two Möbius mappings, Δ_B and Δ_C their determinants, then $\Delta_{BC} = \Delta_B \Delta_C$ is the determinant of the Möbius mapping BC.

Proof. By the definition of group (see Section 44. Algebra and Groups), we need to show that the composition of two Möbius transformations is again a Möbius transformation (so that function composition is a binary operation on the set of Möbius transformations), associativity holds, there is an identity, and each Möbius transformation has an inverse Möbius transformation.

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Let $B : z' = \dfrac{az+b}{cz+d}$ and $C : z' = \dfrac{a'z+b'}{c'z+d'}$ be Möbius transformations, so that $\Delta_B = ad - bc$ and $\Delta_C = a'd' - b'c'$ are nonzero.

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Theorem 52.5 (continued 1)

Proof (continued). Then the composition of B and C is

$$BC : z' = \frac{a'(az+b)/(cz+d)+b'}{c'(az+b)/(cz+d)+d'}$$

$$= \frac{a'(az+b)+b'(cz+d)}{c'(az+b)+d'(cz+d)} = \frac{(aa'+b'c)z+a'b+b'd}{(ac'+cd')z+bc'+dd'},$$

and

$$\begin{aligned}\Delta_{BC} &= (aa'+b'c)(bc'+dd')-(a'b+b'd)(ac'+cd')\\ &= aa'bc'+aa'dd'+bb'cc'+b'cdd'\\ &\quad -aa'bc-a'bcd'-ab'c'd-b'cdd'\\ &= aa'dd'+bb'cc'-ab'c'd-a'bcd'\\ &= (ad-bc)(a'd'-b'c') = \Delta_B\Delta_C \neq 0.\end{aligned}$$

Since $\Delta_{BC} \neq 0$, then the composition BC is also a Möbius transformation.

Theorem 52.5 (continued 2)

Theorem 52.5. Möbius transformations form a group $\mathscr{B}$ under composition of mappings. If B and C are two Möbius mappings, Δ_B and Δ_C their determinants, then $\Delta_{BC} = \Delta_B \Delta_C$ is the determinant of the Möbius mapping BC.

Proof (continued). Since function composition is associative (in general), then associativity holds. The identity Möbius transformation is $z' = z$. We saw in Note 52.A that every Möbius transformation has an inverse.

Therefore the definition of "group" is satisfied and the claim holds. $\square$