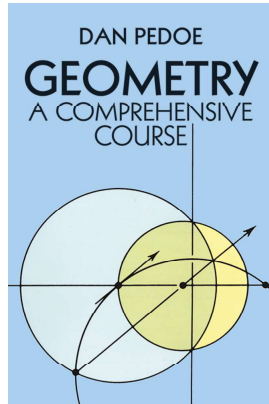


Real Analysis

Chapter VI. Mappings of the Inversive Plane 53. Cross-Ratio—Proofs of Theorems



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Theorem 53.1

Theorem 53.1

Theorem 53.1. If a Möbius transformation has more than two distinct fixed points, then it is the identity mapping, $z' = z$.

Proof. Let w be a fixed point of Möbius transformation $z' = \frac{az + b}{cz + d}$.

Then $w = \frac{aw + b}{cw + d}$ and $cw^2 + (d - a)w - b = 0$. Unless this quadratic equation reduces to the identity $0 = 0$, then there are at most two possible values of w (i.e., at most two possible fixed points). If the quadratic equation *does* reduce to $0 = 0$, then we must have $c = b = d - a = 0$ so that $z' = \frac{az}{a} = z$ is the identity mapping (notice that $\Delta = ad - bc = ad \neq 0$ here, so we do not have $a = d = 0$). So if the transformation fixes more than two distinct points then it is the identity, as claimed. \square

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Theorem 53.2

Theorem 53.2

Theorem 53.2. A Möbius transformation is uniquely determined by the assignment of three distinct points z_j and their three distinct image points z'_j ($j = 1, 2, 3$).

Proof. Consider the transformation $B : z' = \frac{(z - z_2)(z_1 - z_3)}{(z - z_1)(z_2 - z_3)}$. The determinant is $\Delta_B = (z_1 - z_3)(-z_1(z_2 - z_3)) - (-z_2(z_1 - z_3))(z_2 - z_3) = (z_1 - z_3)(z_2 - z_3)(z_2 - z_1) \neq 0$ since z_1, z_2, z_3 are distinct complex numbers, and so B is a Möbius transformation. Notice that $B(z_1) = \infty$, $B(z_2) = 0$, and $B(z_3) = 1$. Now define $C : z' = \frac{(z - z'_2)(z'_1 - z'_3)}{(z - z'_1)(z'_2 - z'_3)}$. Similar to above, the determinant is $\Delta_C = (z'_1 - z'_3)(z'_2 - z'_3)(z'_2 - z'_1) \neq 0$ and so C is a Möbius transformation. Notice that $C(z'_1) = \infty$, $C(z'_2) = 0$, and $C(z'_3) = 1$.

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Theorem 53.2

Theorem 53.2 (continued)

Theorem 53.2. A Möbius transformation is uniquely determined by the assignment of three distinct points z_j and their three distinct image points z'_j ($j = 1, 2, 3$).

Proof (continued). Now $C^{-1} \circ B = C^{-1}B$ is a Möbius transformation satisfying $(C^{-1}B)(z_1) = C^{-1}(B(z_1)) = C^{-1}(\infty) = z'_1$, $(C^{-1}B)(z_2) = C^{-1}(B(z_2)) = C^{-1}(0) = z'_2$, and $(C^{-1}B)(z_3) = C^{-1}(B(z_3)) = C^{-1}(1) = z'_3$. Suppose A is any Möbius transformation that also satisfies $A(z_1) = z'_1$, $A(z_2) = z'_2$, and $A(z_3) = z'_3$. Then $A^{-1} \circ (C^{-1}B) = A^{-1}C^{-1}B$ fixes the distinct points z_1, z_2 , and z_3 . Therefore, by Theorem 53.1, $A^{-1}C^{-1}B$ is the identity transformation and hence $A = C^{-1}B$ (this follows from the fact that an inverse transformation is unique; we can base this on properties of a group and Theorem 52.5). Therefore there is only one transformation mapping z_j to z'_j , respectively, for $j = 1, 2, 3$, as claimed. (This transformation is $C^{-1}B$.) \square

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Theorem 53.3

Theorem 53.3. The cross-ratio satisfies $(z_1, z_2; z_3, z_4) = (z_2, z_1; z_4, z_3)$.

Proof. By definition,

$$\begin{aligned} (z_1, z_2; z_3, z_4) &= \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)} \\ &= \frac{(z_2 - z_4)(z_1 - z_3)}{(z_1 - z_4)(z_2 - z_3)} = (z_2, z_1; z_4, z_3), \end{aligned}$$

as claimed. \square

Theorem 52.4

Theorem 53.4. The cross-ratio of four points is an invariant under Möbius transformations.

Proof. Let A be a Möbius transformation such that $A(z_j) = z'_j$ for $j = 1, 2, 3$. In the proof of Theorem 53.2, we saw for Möbius transformations B and C , where $B : z' = \frac{(z - z_2)(z_1 - z_3)}{(z - z_1)(z_2 - z_3)}$ and $C : z' = \frac{(z - z'_2)(z'_1 - z'_3)}{(z - z'_1)(z'_2 - z'_3)}$, that $A = C^{-1} \circ B = C^{-1}B$. So $A(z) = (C^{-1}B)(z) = z'$ for all $z \in \mathbb{C}$ and, in particular, with $z = z_4$ we have $(C^{-1}B)(z_4) = z'_4$ or $B(z_4) = C(z'_4)$. This gives

$$B(z_4) = \frac{((z_4) - z_2)(z_1 - z_3)}{((z_4) - z_1)(z_2 - z_3)} = \frac{((z'_4) - z'_2)(z'_1 - z'_3)}{((z'_4) - z'_1)(z'_2 - z'_3)} = C(z'_4),$$

or $(z_1, z_2; z_3, z_4) = (z'_1, z'_2; z'_3, z'_4)$, as claimed. \square

Theorem 52.5

Theorem 53.5. The cross-ratio of four points is real if and only if the four points lie on a straight line or a circle.

Proof. Let the four distinct points by z_j for $j = 1, 2, 3, 4$. First, suppose the cross-ratio $(z_1, z_2; z_3, z_4) = k$ is real. Let z'_1, z'_2, z'_3 be three distinct points on the real axis. Let T be the unique Möbius transformation which maps z_1, z_2, z_3 onto the three distinct points z'_1, z'_2, z'_3 , respectively (guaranteed to exist by Theorem 53.2). Let $z'_4 = T(z_4)$. By Theorem 53.4 (invariance of the cross-ratio under a Möbius transformation),

$$k = \frac{(z_4 - z_2)(z_1 - z_3)}{(z_4 - z_1)(z_2 - z_3)} = \frac{(z'_4 - z'_2)(z'_1 - z'_3)}{(z'_4 - z'_1)(z'_2 - z'_3)}.$$

Hence $(z'_4 - z'_2)/(z'_4 - z'_1) = k(z'_2 - z'_3)/(z'_1 - z'_3)$ is real (since z'_1, z'_2, z'_3 are real), and solving this equation (linear in z'_4) for z'_4 , we see that z'_4 is real.

Theorem 52.5 (continued)

Proof (continued). So z'_1, z'_2, z'_3, z'_4 all lie on the real axis (a line). Now T^{-1} is a Möbius transformation, so by Theorem 52.3 T^{-1} maps the real axis to a circle or line. Since T^{-1} maps z'_1, z'_2, z'_3, z'_4 to z_1, z_2, z_3, z_4 , respectively, then it must be that z_1, z_2, z_3, z_4 lie on a straight line or circle when $(z_1, z_2; z_3, z_4)$ is real, as claimed.

Now suppose the four given points z_1, z_2, z_3, z_4 lie on a line or a circle. Let z'_1, z'_2, z'_3 be distinct points on the real axis. Then by Theorem 53.2 there is a unique Möbius transformation T which maps z_j to z'_j for $j = 1, 2, 3$. Since T maps a circle or line to a circle or line by Theorem 52.3, then $T(z_1) = z'_1$, $T(z_2) = z'_2$, $T(z_3) = z'_3$ lie on a line or circle; by choice, z'_1, z'_2, z'_3 lie on the real axis so it must be that z'_4 also lies on the real axis. Since z'_1, z'_2, z'_3, z'_4 are all real, then the cross-ratio $(z'_1, z'_2; z'_3, z'_4)$ is real. Since the cross-ratio is invariant under Möbius transformation T^{-1} , then we must also have $(z_1, z_2; z_3, z_4)$ is real, as claimed. \square