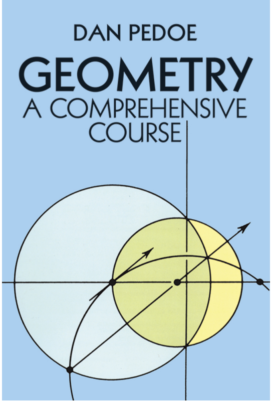
Real Analysis

Chapter VI. Mappings of the Inversive Plane

53. Cross-Ratio—Proofs of Theorems



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Proof. Let w be a fixed point of Möbius transformation $z' = \dfrac{az+b}{cz+d}$. Then $w = \dfrac{aw+b}{cw+d}$ and $cw^2 + (d-a)w - b = 0$. Unless this quadratic equation reduces to the identity $0 = 0$, then there are at most two possible values of w (i.e., at most two possible fixed points).

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Theorem 53.2

Theorem 53.2. A Möbius transformation is uniquely determined by the assignment of three distinct points z_j and their three distinct image points z_j' ($j = 1, 2, 3$).

Proof. Consider the transformation $B : z' = \dfrac{(z - z_2)(z_1 - z_3)}{(z - z_1)(z_2 - z_3)}$. The determinant is $\Delta_B = (z_1 - z_3)(-z_1(z_2 - z_3)) - (-z_2(z_1 - z_3))(z_2 - z_3) = (z_1 - z_3)(z_2 - z_3)(z_2 - z_1) \neq 0$ since z_1, z_2, z_3 are distinct complex numbers, and so B is a Möbius transformation. Notice that $B(z_1) = \infty$, $B(z_2) = 0$, and $B(z_3) = 1$.

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Theorem 53.2 (continued)

Theorem 53.2. A Möbius transformation is uniquely determined by the assignment of three distinct points z_j and their three distinct image points z_j' $(j = 1, 2, 3)$.

Proof (continued). Now $C^{-1} \circ B = C^{-1}B$ is a Möbius transformation satisfying $(C^{-1}B)(z_1) = C^{-1}(B(z_1)) = C^{-1}(\infty) = z_1'$,
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$(C^{-1}B)(z_3) = C^{-1}(B(z_3)) = C^{-1}(1) = z_3'$. Suppose A is any Möbius transformation that also satisfies $A(z_1) = z_1'$, $A(z_2) = z_2'$, and $A(z_3) = z_3'$. Then $A^{-1} \circ (C^{-1}B) = A^{-1}C^{-1}B$ fixes the distinct points z_1, z_2, and z_3. Therefore, by Theorem 53.1, $A^{-1}C^{-1}B$ is the identity transformation and hence $A = C^{-1}B$ (this follows from the fact that an inverse transformation is unique; we can base this on properties of a group and Theorem 52.5). Therefore there is only one transformation mapping z_j to z_j', respectively, for $j = 1, 2, 3$, as claimed. (This transformation is $C^{-1}B$.) □

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Theorem 53.3

Theorem 53.3. The cross-ratio satisfies $(z_1, z_2; z_3, z_4) = (z_2, z_1; z_4, z_3)$.

Proof. By definition,

$$(z_1, z_2; z_3, z_4) = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_2 - z_3)(z_1 - z_4)}$$

$$= \frac{(z_2 - z_4)(z_1 - z_3)}{(z_1 - z_4)(z_2 - z_3)} = (z_2, z_1; z_4, z_3),$$

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Theorem 52.4

Theorem 53.4. The cross-ratio of four points is an invariant under Möbius transformations.

Proof. Let A be a Möbius transformation such that $A(z_j) = z_j'$ for $j = 1, 2, 3$. In the proof of Theorem 53.2, we saw for Möbius transformations B and C, where $B : z' = \dfrac{(z - z_2)(z_1 - z_3)}{(z - z_1)(z_2 - z_3)}$ and $C : z' = \dfrac{(z - z_2')(z_1' - z_3')}{(z - z_1')(z_2' - z_3')}$, that $A = C^{-1} \circ B = C^{-1}B$.

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$$B(z_4) = \frac{((z_4) - z_2)(z_1 - z_3)}{((z_4) - z_1)(z_2 - z_3)} = \frac{((z_4') - z_2')(z_1' - z_3')}{((z_4') - z_1')(z_2' - z_3')} = C(z_4'),$$

or $(z_1, z_2; z_3, z_4) = (z_1', z_2'; z_3', z_4')$, as claimed. □

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Theorem 52.5

Theorem 53.5. The cross-ratio of four points is real if and only if the four points lie on a straight line or a circle.

Proof. Let the four distinct points by z_j for $j = 1, 2, 3, 4$. First, suppose the cross-ratio $(z_1, z_2; z_3, z_4) = k$ is real. Let z_1', z_2', z_3' be three distinct points on the real axis. Let T by the unique Möbius transformation which maps z_1, z_2, z_3 onto the three distinct points z_1', z_2', z_3', respectively (guaranteed to exist by Theorem 53.2). Let $z_4' = T(z_4)$.

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$$k = \frac{(z_4 - z_2)(z_1 - z_3)}{(z_4 - z_1)(z_2 - z_3)} = \frac{(z_4' - z_2')(z_1' - z_3')}{(z_4' - z_1')(z_2' - z_3')}.$$

Hence $(z_4' - z_2')/(z_4' - z_1') = k(z_2' - z_3')/(z_1' - z_3')$ is real (since z_1', z_2', z_3' are real), and solving this equation (linear in z_4') for z_4', we see that z_4' is real.

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Theorem 52.5 (continued)

Proof (continued). So z_1', z_2', z_3', z_4' all lie on the real axis (a line). Now T^{-1} is a Möbius transformation, so by Theorem 52.3 T^{-1} maps the real axis to a circle or line. Since T^{-1} maps z_1', z_2', z_3', z_4' to z_1, z_2, z_3, z_4, respectively, then it must be that z_1, z_2, z_3, z_4 lie on a straight line or circle when $(z_1, z_2; z_3, z_4)$ is real, as claimed.

Now suppose the four given points z_1, z_2, z_3, z_4 lie on a line or a circle. Let z_1', z_2', z_3' be distinct points on the real axis. Then by Theorem 53.2 there is a unique Möbius transformation T which maps z_j to z_j' for $j = 1, 2, 3$. Since T maps a circle of line to a circle or line by Theorem 52.3, then $T(z_1) = z_1'$, $T(z_2) = z_2'$, $T(z_3) = z_3'$ lie on a line or circle; by choice, z_1', z_2', z_3' lie on the real axis so it must be that z_4' also lies on the real axis.

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