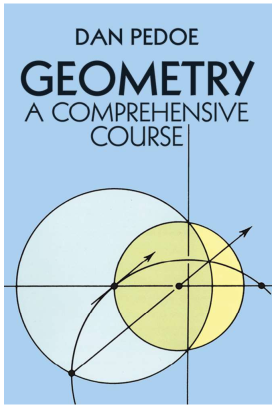
Real Analysis

Chapter VI. Mappings of the Inversive Plane
54. Circles and Conformality—Proofs of Theorems



Theorem 54.1

Theorem 54.1. If a circle $\mathscr{C}$ is mapped onto a circle $\mathscr{C}'$ by a Möbius transformation, then the interior of $\mathscr{C}$ is mapped onto the interior of $\mathscr{C}$, or onto the exterior of $\mathscr{C}'$.

Proof. By Theorem 52.A (see the proof), a Möbius transformation may be written as the composition of a translation, and inversion in the unit circle followed by a reflection in the real axis. With the exception of the inversion ($z' = 1/z$), these mappings map the inside of an "object circle" onto the inside of the "image circle." Inversion may map the inside of the object circle onto the inside of the image circle, or it may map the inside of the object circle onto the outside of image circle. We see by considering the modulus of z and z', that the inversion $z' = 1/z$ maps the interior of the object circle to the exterior of the image circle, and maps the exterior of the object circle to the interior of the image circle.

Theorem 54.1 (continued)

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Proof (continued). So if an inversion is not present in the dissection of the Möbius transformation given by Theorem 52.A (this occurs when $c = 0$ in the usual notation for the transformation), then the interior of $\mathscr{C}$ is mapped to the interior of $\mathscr{C}'$. If an inversion is present in the dissection (which occurs when $c \neq 0$), then the interior of $\mathscr{C}$ is mapped to the interior of $\mathscr{C}'$. □

Theorem 54.A

Theorem 54.A. A Möbius transformation of the form $Z = \dfrac{az+b}{\overline{b}z+\overline{a}}$ where $|b| < |b|$ maps the unit circle $|z| = 1$ to itself and maps the interior of $|z| = 1$ to itself.

Proof. Consider the "object circle" $|z| = 1$ and its "image circle" $|Z| = 1$ under Möbius transformation $Z = \dfrac{az+b}{cz+d}$ which maps the interior $|z| < 1$ to the interior $|Z| < 1$. For any $|z|^2 = z\overline{z} = 1$, we need the corresponding value Z to satisfy the condition $|Z|^2 = Z\overline{Z} = 1$. So for $|z| = 1$, we require

$$\frac{(az+b)}{(cz+d)}\frac{\overline{(az+b)}}{\overline{(cz+d)}} - 1 = 0,$$

which is equivalent to $(az+b)\overline{(az+b)} - (cz+d)\overline{(cz+d)} = 0$ or $a\overline{a}z\overline{z} + a\overline{b}z + \overline{a}b\overline{z} + b\overline{b} - c\overline{c}z\overline{z} - c\overline{d}z - \overline{c}d\overline{z} - d\overline{d} = 0$ or $z\overline{z}(a\overline{a} - c\overline{c}) + z(a\overline{b} + c\overline{d}) + \overline{z}(\overline{a}b - \overline{c}d) + b\overline{b} - d\overline{d} = 0$.

Theorem 54.A (continued 1)

Proof (continued).
$\ldots z\bar{z}(a\bar{a} - c\bar{c}) + z(a\bar{b} + c\bar{d}) + \bar{z}(\bar{a}b - \bar{c}d) + b\bar{b} - d\bar{d} = 0$. Since we have $z\bar{z} = 1$, then this simplifies to
$z(a\bar{b} + c\bar{d}) + \bar{z}(\bar{a}b - \bar{c}d) + (a\bar{a} - c\bar{c}) + b\bar{b} - d\bar{d} = 0$. Since this must hold for all z with $|z| = 1$, then we need the coefficients of z and $\bar{z}$ are equal to 0, and the constant is equal to 0:

$$a\bar{b} + c\bar{d} = \bar{a}b - \bar{c}d = 0 \text{ and } a\bar{a} - c\bar{c} + b\bar{b} - d\bar{d} = 0.$$

Since $\overline{\bar{a}b - \bar{c}d} = a\bar{b} + c\bar{d}$ and $a\bar{a} - c\bar{c} + b\bar{b} - d\bar{d} = |a|^2 + |b|^2 - |c|^2 - |d|^2$, then this reduces to two equations:

$$a\bar{b} + c\bar{d} = 0 \text{ and } |a|^2 + |b|^2 = |c|^2 + |d|^2.$$

Set $k = c/\bar{b}$ so that $c = k\bar{b}$. Then condition $a\bar{b} + c\bar{d} = 0$ is equivalent to $\bar{k}bd - \bar{a}b = 0$ so that $d = \bar{a}/\bar{k}$. The condition $|a|^2 + |b|^2 = |c|^2 + |d|^2$ then implies that $|k|^2/|b|^2 + |a|^2/|k|^2 = |a|^2 + |b|^2$ so that we *can* take $|k| = 1$.

Theorem 54.A (continued 2)

Proof (continued). Then $1/\bar{k} = k$ and

$$Z = \frac{az + b}{cz + d} = \frac{az + b}{k(\bar{b}z + \bar{a})}, \text{ where } |k| = 1,$$

or equivalently $Z = k\dfrac{az + b}{\bar{b}z + \bar{a}}$, where $|k| = 1$.

This transformation maps $|z| = 1$ to $|Z| = 1$. Now we need to deal with the "interior" property. The determinant of the transformation is $\Delta = k(|a|^2 - |b|^2)$. If we *choose* $k = 1$ and $\Delta = 1$ (at this stage we no longer have the most general version of such a transformation), the $|a|^2 - |b|^2 = 1$ and so $|b| < |a|$. The point 0 (inside $|z| = 1$) is then mapped to the point $b/\bar{a}$ and since $|b| < |a|$ then this point is inside the circle $|Z| = 1$. Therefore the transformation $Z = \dfrac{az + b}{\bar{b}z + \bar{a}}$ is a Möbius transformation which maps the circle $|z| = 1$ to the circle $|Z| = 1$ and maps the inside of $|z| = 1$ to the inside of $|Z| = 1$, as claimed. □

Theorem 54.B

Theorem 54.B. The set of all Möbius transformations transformations mapping $|z| = 1$ to itself and mapping the interior of $|z| = 1$ to itself forms a subgroup of the group $\mathscr{B}$ of all Möbius transformations.

Proof. To show that a nonempty subset of a group is a subgroup, it suffices to show that for all A, C in the set we have AC^{-1} in the set (see my online notes for Modern Algebra 1 [MATH 5410] on Section I.2. Homomorphisms and Subgroups; notice Theorem I.2.5). Notice that for $Z = e^{i\theta}\left(\dfrac{z - \alpha}{\bar{\alpha}z - 1}\right) = \dfrac{e^{i\theta}z - \alpha e^{i\theta}}{\bar{\alpha}z - 1}$ in the set, we have that the inverse

$$z = \frac{-(-1)Z + (-\alpha e^{i\theta})}{(\bar{\alpha})Z - (e^{i\theta})} = \frac{Z - \alpha e^{i\theta}}{\bar{\alpha}Z - e^{i\theta}} = e^{i(-\theta)}\left(\frac{Z - \alpha e^{i\theta}}{\overline{\alpha e^{i\theta}}Z - 1}\right)$$

is also in the set. That is, the set is closed under inverses.

Theorem 54.B (continued 1)

Proof (continued). Next, if $A(z) = e^{i\theta_1}\left(\dfrac{z - \alpha}{\bar{\alpha}z - 1}\right)$ and $B(z) = e^{i\theta_2}\left(\dfrac{z - \beta}{\bar{\beta}z - 1}\right)$ are in the set then the composition is

$$(AB)(z) = A(B(z)) = A\left(e^{i\theta_2}\left(\frac{z - \beta}{\bar{\beta}z - 1}\right)\right) = e^{i\theta_1}\left(\frac{e^{i\theta_2}\left(\frac{z-\beta}{\bar{\beta}z-1}\right) - \alpha}{\bar{\alpha}e^{i\theta_2}\left(\frac{z-\beta}{\bar{\beta}z-1}\right) - 1}\right)$$

$$= e^{i\theta_1}\left(\frac{e^{i\theta_2}(z - \beta) - \alpha(\bar{\beta}z - 1)}{\bar{\alpha}e^{i\theta_2}(z - \beta) - (\bar{\beta}z - 1)}\right) = e^{i\theta_1}\left(\frac{(e^{i\theta_2} - \alpha\bar{\beta})z - \beta e^{i\theta_2} + \alpha}{(\bar{\alpha}e^{i\theta_2} - \bar{\beta})z - \bar{\alpha}\beta e^{i\theta_2} + 1}\right)$$

$$= \frac{e^{i\theta_1}}{-e^{i\theta_2}}\left(\frac{(e^{i\theta_2} - \alpha\bar{\beta})z - (\beta e^{i\theta_1} - \alpha)}{(\bar{\beta}e^{-i\theta_2} - \bar{\alpha})z - (e^{-i\theta_2} - \bar{\alpha}\beta)}\right) = e^{i(\theta_1 - \theta_2 + \pi)}\left(\frac{\gamma z - \delta}{\bar{\delta}z - \bar{\gamma}}\right)$$

where $\gamma = e^{i\theta_2} - \alpha\bar{\beta}$ and $\delta = \beta e^{i\theta_2} - \alpha$.

Theorem 54.B (continued 2)

Proof (continued). Define θ_3 as the argument of $\overline{\gamma}/\gamma$, so that $e^{-i\theta_3}\overline{\gamma}/\gamma = 1$. Then

$$(AB)(z) = e^{i(\theta_1-\theta_2+\pi)}\left(e^{-i\theta_3}\frac{\overline{\gamma}}{\gamma}\right)\left(\frac{\gamma z-\delta}{\overline{\delta}z-\overline{\gamma}}\right) = e^{i(\theta_1-\theta_2-\theta_3+\pi)}\left(\frac{\gamma\overline{\gamma}z-\delta\overline{\gamma}}{\overline{\delta}\gamma z-\gamma\overline{\gamma}}\right)$$

$$= e^{i(\theta_1-\theta_2-\theta_3+\pi)}\frac{|\gamma|^2z-\delta\overline{\gamma}}{\overline{\delta}\gamma z-|\gamma|^2} = e^{i(\theta_1-\theta_2-\theta_3+\pi)}\frac{z-(\delta\overline{\gamma}/|\gamma|^2)}{(\overline{\delta\overline{\gamma}/|\gamma|^2})z-1} = e^{i\theta}\left(\frac{z-\alpha}{\overline{\alpha}z-1}\right)$$

where $\theta = \theta_1-\theta_2-\theta_3+\pi$ and $\alpha = \delta\overline{\gamma}/|\gamma|^2$. Since 0 is interior to $|z|=1$ then $B(0)$ is interior to $|z|=1$. Since $B(0)$ is is interior to $|z|=1$ then $A(B(0)) = e^{i\theta}\delta\overline{\gamma}/|\gamma|^2$ is interior to $|z|=1$. That is, $|e^{i\theta}\delta\overline{\gamma}/|\gamma|^2| = |\delta\overline{\gamma}/|\gamma|^2| = |\alpha| < 1$. Hence, AB is in the set of transformations and the set is closed under function composition. So for an A, C in the set we have AC^{-1} in the set, and by the theorem mentioned above we have that the set is a subgroup of $\mathscr{B}$, as claimed. □