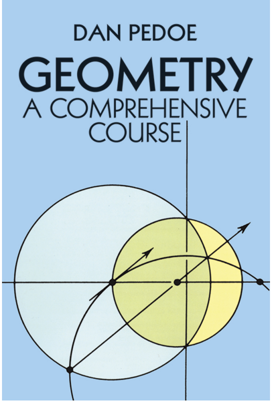
Real Analysis

Chapter VI. Mappings of the Inversive Plane

54. Circles and Conformality—Proofs of Theorems



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Proof. By Theorem 52.A (see the proof), a Möbius transformation may be written as the composition of a translation, and inversion in the unit circle followed by a reflection in the real axis. With the exception of the inversion ($z' = 1/z$), these mappings map the inside of an "object circle" onto the inside of the "image circle."

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Theorem 54.1 (continued)

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Proof (continued). So if an inversion is not present in the dissection of the Möbius transformation given by Theorem 52.A (this occurs when $c = 0$ in the usual notation for the transformation), then the interior of $\mathscr{C}$ is mapped to the interior of $\mathscr{C}'$. If an inversion is present in the dissection (which occurs when $c \neq 0$), then the interior of $\mathscr{C}$ is mapped to the interior of $\mathscr{C}'$. □

Theorem 54.A

Theorem 54.A. A Möbius transformation of the form $Z = \dfrac{az+b}{\overline{b}z+\overline{a}}$ where $|b| < |b|$ maps the unit circle $|z| = 1$ to itself and maps the interior of $|z| = 1$ to itself.

Proof. Consider the "object circle" $|z| = 1$ and its "image circle" $|Z| = 1$ under Möbius transformation $Z = \dfrac{az+b}{cz+d}$ which maps the interior $|z| < 1$ to the interior $|Z| < 1$. For any $|z|^2 = z\overline{z} = 1$, we need the corresponding value Z to satisfy the condition $|Z|^2 = Z\overline{Z} = 1$. So for $|z| = 1$, we require

$$\frac{(az+b)}{(cz+d)}\frac{\overline{(az+b)}}{\overline{(cz+d)}} - 1 = 0,$$

which is equivalent to $(az+b)\overline{(az+b)} - (cz+d)\overline{(cz+d)} = 0$ or $a\overline{a}z\overline{z} + a\overline{b}z + \overline{a}b\overline{z} + b\overline{b} - c\overline{c}z\overline{z} - c\overline{d}z - \overline{c}d\overline{z} - d\overline{d} = 0$ or $z\overline{z}(a\overline{a} - c\overline{c}) + z(a\overline{b} + c\overline{d}) + \overline{z}(\overline{a}b - \overline{c}d) + b\overline{b} - d\overline{d} = 0.$

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Theorem 54.A (continued 1)

Proof (continued).
$\ldots z\overline{z}(a\overline{a} - c\overline{c}) + z(a\overline{b} + c\overline{d}) + \overline{z}(\overline{a}b - \overline{c}d) + b\overline{b} - d\overline{d} = 0$. Since we have $z\overline{z} = 1$, then this simplifies to $z(a\overline{b} + c\overline{d}) + \overline{z}(\overline{a}b - \overline{c}d) + (a\overline{a} - c\overline{c}) + b\overline{b} - d\overline{d} = 0$. Since this must hold for all z with $|z| = 1$, then we need the coefficients of z and $\overline{z}$ are equal to 0, and the constant is equal to 0:

$$a\overline{b} + c\overline{d} = \overline{a}b - \overline{c}d = 0 \text{ and } a\overline{a} - c\overline{c} + b\overline{b} - d\overline{d} = 0.$$

Since $\overline{\overline{a}b - \overline{c}d} = a\overline{b} + c\overline{d}$ and $a\overline{a} - c\overline{c} + b\overline{b} - d\overline{d} = |a|^2 + |b|^2 - |c|^2 - |d|^2$, then this reduces to two equations:

$$a\overline{b} + c\overline{d} = 0 \text{ and } |a|^2 + |b|^2 = |c|^2 + |d|^2.$$

Set $k = c/\overline{b}$ so that $c = k\overline{b}$. Then condition $a\overline{b} + c\overline{d} = 0$ is equivalent to $\overline{k}bd - \overline{a}b = 0$ so that $d = \overline{a}/\overline{k}$. The condition $|a|^2 + |b|^2 = |c|^2 + |d|^2$ then implies that $|k|^2/|b|^2 + |a|^2/|k|^2 = |a|^2 + |b|^2$ so that we *can* take $|k| = 1$.

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$\ldots z\overline{z}(a\overline{a} - c\overline{c}) + z(a\overline{b} + c\overline{d}) + \overline{z}(\overline{a}b - \overline{c}d) + b\overline{b} - d\overline{d} = 0$. Since we have $z\overline{z} = 1$, then this simplifies to
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Since $\overline{\overline{a}b - \overline{c}d} = a\overline{b} + c\overline{d}$ and $a\overline{a} - c\overline{c} + b\overline{b} - d\overline{d} = |a|^2 + |b|^2 - |c|^2 - |d|^2$, then this reduces to two equations:

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Theorem 54.A (continued 2)

Proof (continued). Then $1/\overline{k} = k$ and

$$Z = \frac{az+b}{cz+d} = \frac{az+b}{k(\overline{b}z+\overline{a})}, \text{ where } |k| = 1,$$

or equivalently $Z = k\dfrac{az+b}{\overline{b}z+\overline{a}}$, where $|k| = 1$.

This transformation maps $|z| = 1$ to $|Z| = 1$. Now we need to deal with the "interior" property. The determinant of the transformation is $\Delta = k(|a|^2 - |b|^2)$. If we *choose* $k = 1$ and $\Delta = 1$ (at this stage we no longer have the most general version of such a transformation), the $|a|^2 - |b|^2 = 1$ and so $|b| < |a|$. The point 0 (inside $|z| = 1$) is then mapped to the point $b/\overline{a}$ and since $|b| < |a|$ then this point is inside the circle $|Z| = 1$. Therefore the transformation $Z = \dfrac{az+b}{\overline{b}z+\overline{a}}$ is a Möbius transformation which maps the circle $|z| = 1$ to the circle $|Z| = 1$ and maps the inside of $|z| = 1$ to the inside of $|Z| = 1$, as claimed. □

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Theorem 54.B

Theorem 54.B. The set of all Möbius transformations transformations mapping $|z| = 1$ to itself and mapping the interior of $|z| = 1$ to itself forms a subgroup of the group $\mathscr{B}$ of all Möbius transformations.

Proof. To show that a nonempty subset of a group is a subgroup, it suffices to show that for all A, C in the set we have AC^{-1} in the set (see my online notes for Modern Algebra 1 [MATH 5410] on Section I.2. Homomorphisms and Subgroups; notice Theorem I.2.5).

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$$z = \frac{-(-1)Z + (-\alpha e^{i\theta})}{(\overline{\alpha})Z - (e^{i\theta})} = \frac{Z - \alpha e^{i\theta}}{\overline{\alpha}Z - e^{i\theta}} = e^{i(-\theta)}\left(\frac{Z - \alpha e^{i\theta}}{\overline{\alpha e^{i\theta}}Z - 1}\right)$$

is also in the set. That is, the set is closed under inverses.

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Theorem 54.B (continued 1)

Proof (continued). Next, if $A(z) = e^{i\theta_1}\left(\frac{z-\alpha}{\overline{\alpha}z-1}\right)$ and $B(z) = e^{i\theta_2}\left(\frac{z-\beta}{\overline{\beta}z-1}\right)$ are in the set then the composition is

$$(AB)(z) = A(B(z)) = A\left(e^{i\theta_2}\left(\frac{z-\beta}{\overline{\beta}z-1}\right)\right) = e^{i\theta_1}\left(\frac{e^{i\theta_2}\left(\frac{z-\beta}{\overline{\beta}z-1}\right)-\alpha}{\overline{\alpha}e^{i\theta_2}\left(\frac{z-\beta}{\overline{\beta}z-1}\right)-1}\right)$$

$$= e^{i\theta_1}\left(\frac{e^{i\theta_2}(z-\beta)-\alpha(\overline{\beta}z-1)}{\overline{\alpha}e^{i\theta_2}(z-\beta)-(\overline{\beta}z-1)}\right) = e^{i\theta_1}\left(\frac{(e^{i\theta_2}-\alpha\overline{\beta})z-\beta e^{i\theta_2}+\alpha}{(\overline{\alpha}e^{i\theta_2}-\overline{\beta})z-\overline{\alpha}\beta e^{i\theta_2}+1}\right)$$

$$= \frac{e^{i\theta_1}}{-e^{i\theta_2}}\left(\frac{(e^{i\theta_2}-\alpha\overline{\beta})z-(\beta e^{i\theta_1}-\alpha)}{(\overline{\beta}e^{-i\theta_2}-\overline{\alpha})z-(e^{-i\theta_2}-\overline{\alpha}\beta)}\right) = e^{i(\theta_1-\theta_2+\pi)}\left(\frac{\gamma z-\delta}{\overline{\delta}z-\overline{\gamma}}\right)$$

where $\gamma = e^{i\theta_2}-\alpha\overline{\beta}$ and $\delta = \beta e^{i\theta_2}-\alpha$.

Theorem 54.B (continued 2)

Proof (continued). Define θ_3 as the argument of $\overline{\gamma}/\gamma$, so that $e^{-i\theta_3}\overline{\gamma}/\gamma = 1$. Then

$$(AB)(z) = e^{i(\theta_1-\theta_2+\pi)}\left(e^{-i\theta_3}\frac{\overline{\gamma}}{\gamma}\right)\left(\frac{\gamma z-\delta}{\overline{\delta}z-\overline{\gamma}}\right) = e^{i(\theta_1-\theta_2-\theta_3+\pi)}\left(\frac{\gamma\overline{\gamma}z-\delta\overline{\gamma}}{\overline{\delta}\gamma z-\gamma\overline{\gamma}}\right)$$

$$= e^{i(\theta_1-\theta_2-\theta_3+\pi)}\frac{|\gamma|^2 z-\delta\overline{\gamma}}{\overline{\delta}\gamma z-|\gamma|^2} = e^{i(\theta_1-\theta_2-\theta_3+\pi)}\frac{z-(\delta\overline{\gamma}/|\gamma|^2)}{\overline{(\delta\overline{\gamma}/|\gamma|^2)}z-1} = e^{i\theta}\left(\frac{z-\alpha}{\overline{\alpha}z-1}\right)$$

where $\theta = \theta_1-\theta_2-\theta_3+\pi$ and $\alpha = \delta\overline{\gamma}/|\gamma|^2$. Since 0 is interior to $|z|=1$ then $B(0)$ is interior to $|z|=1$. Since $B(0)$ is is interior to $|z|=1$ then $A(B(0)) = e^{i\theta}\delta\overline{\gamma}/|\gamma|^2$ is interior to $|z|=1$. That is, $|e^{i\theta}\delta\overline{\gamma}/|\gamma|^2| = |\delta\overline{\gamma}/|\gamma|^2| = |\alpha| < 1$. Hence, AB is in the set of transformations and the set is closed under function composition. So for an A, C in the set we have AC^{-1} in the set, and by the theorem mentioned above we have that the set is a subgroup of $\mathscr{B}$, as claimed. □

Theorem 54.B (continued 2)

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$$(AB)(z) = e^{i(\theta_1-\theta_2+\pi)}\left(e^{-i\theta_3}\frac{\overline{\gamma}}{\gamma}\right)\left(\frac{\gamma z-\delta}{\overline{\delta}z-\overline{\gamma}}\right) = e^{i(\theta_1-\theta_2-\theta_3+\pi)}\left(\frac{\gamma\overline{\gamma}z-\delta\overline{\gamma}}{\overline{\delta}\gamma z-\gamma\overline{\gamma}}\right)$$

$$= e^{i(\theta_1-\theta_2-\theta_3+\pi)}\frac{|\gamma|^2 z-\delta\overline{\gamma}}{\overline{\delta}\gamma z-|\gamma|^2} = e^{i(\theta_1-\theta_2-\theta_3+\pi)}\frac{z-(\delta\overline{\gamma}/|\gamma|^2)}{\overline{(\delta\overline{\gamma}/|\gamma|^2)}z-1} = e^{i\theta}\left(\frac{z-\alpha}{\overline{\alpha}z-1}\right)$$

where $\theta = \theta_1-\theta_2-\theta_3+\pi$ and $\alpha = \delta\overline{\gamma}/|\gamma|^2$. Since 0 is interior to $|z|=1$ then $B(0)$ is interior to $|z|=1$. Since $B(0)$ is is interior to $|z|=1$ then $A(B(0)) = e^{i\theta}\delta\overline{\gamma}/|\gamma|^2$ is interior to $|z|=1$. That is, $|e^{i\theta}\delta\overline{\gamma}/|\gamma|^2| = |\delta\overline{\gamma}/|\gamma|^2| = |\alpha| < 1$. Hence, AB is in the set of transformations and the set is closed under function composition. So for an A, C in the set we have AC^{-1} in the set, and by the theorem mentioned above we have that the set is a subgroup of $\mathscr{B}$, as claimed. □