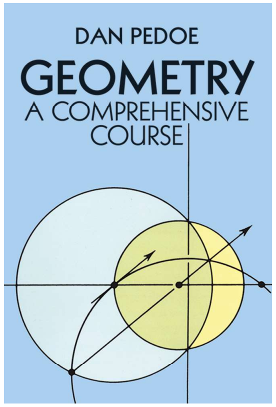
Real Analysis

Chapter VI. Mappings of the Inversive Plane

55. *M*-Transformations—Proofs of Theorems



Theorem 55.2

Theorem 55.2. Let X and A be two points of Ω and let ξ and α be directions through X and A, respectively. Then there is a unique M-transformation which maps X on A, and maps the direction ξ onto the direction α.

Proof. First, by Exercise 55.1 there is exactly one circle orthogonal to ω that passes through point X and has vector ξ as its direction vector at X (Pedoe describes the direction-condition in terms of tangency to the [Euclidean] straight line determined by vector ξ.) Denote this circle as $\mathscr{C}_X$, and let X_1 and X_2 be the points of intersection of $\mathscr{C}_X$ and ω. Similarly, let $\mathscr{C}_A$ be the unique circle through point A with direction α at A and orthogonal to ω. Let A_1 and A_2 be the points of intersection of $\mathscr{C}_X$ with ω. Since Möbius transformations preserve angles of intersection by Theorem 54.3, and an M-transformation maps ω onto itself, circle orthogonal to ω are mapped onto circles orthogonal to ω.

Theorem 55.2 (continued 1)

Proof (continued). Now any M-transformation which maps point A and vector ξ onto point A and vector α, must map $\mathscr{C}_X$ onto $\mathscr{C}_A$ since $\mathscr{C}_X$ is mapped onto a circle through A with direction α at A and orthogonal to ω, and there is only one such circle (by Exercise 55.1), namely $\mathscr{C}_A$.

Next label the points of intersection of $\mathscr{C}_X$ and $\mathscr{C}_A$ with ω, such that the direction ξ of $\mathscr{C}_X$ is that of X_1 towards X_2 and such that the direction α of $\mathscr{C}_A$ is that of A_1 towards A_2 (direction can be put on a line or circle by giving it parametrically; this is also how tangent vectors such as ξ and α can be determined by taking derivatives with respect to the parameter). The M-transformation must map X_1 to A_1, map X_2 to A_2, and map X to A (again, this follows from a parametric presentation of the circle and tangent vectors). By Theorem 53.2, three points and their images uniquely determine a Möbius transformation. We just need to show that this unique Möbius transformation is in fact an M-transformation.

Theorem 55.2 (continued 2)

Theorem 55.2. Let X and A be two points of Ω and let ξ and α be directions through X and A, respectively. Then there is a unique M-transformation which maps X on A, and maps the direction ξ onto the direction α.

Proof (continued). Now the Möbius transformation maps the circle through X_1 and X_2 which is orthogonal to $\mathscr{C}_X$ (namely, circle ω) must be mapped to the circle through A_1 and A_2 which is orthogonal to $\mathscr{C}_A$ (also circle ω; the preservation of orthogonality is given by Theorem 54.3). That is, ω is mapped to ω by the Möbius transformation. Since the interior point X of Ω is mapped to the interior point A of Ω, then the interior of ω is mapped to the interior of ω; that is, the unique Möbius transformation is an M-transformation, as needed. □

Corollary 55.A

Corollary 55.A. Let $\mathscr{C}_A$ be a circle through point A of Ω to ω, and let $\mathscr{C}_B$ be a circle through a point B of Ω orthogonal to ω. Then there exist just two M-transformations which map A on B and $\mathscr{C}_A$ on $\mathscr{C}_B$.

Proof. For a given direction tangent to $\mathscr{C}_A$ at point A and a given tangent to $\mathscr{C}_B$ at point B there is, by Theorem 55.2, a unique M-transformation mapping A to X and mapping $\mathscr{C}_A$ to $\mathscr{C}_B$. Since there are two directions tangent to $\mathscr{C}_B$ at B then for a given direction tangent to $\mathscr{C}_A$ at A there are two such M-transformations (reversing the direction of the tangent to $\mathscr{C}_A$ at A results in the same two M-transformations). Therefore, there are two such M-transformations, as claimed. □

Theorem 55.3

Theorem 55.3. There exists a unique M-transformation which interchanges two given points A and B of Ω.

Proof. Let $\mathscr{C}$ be the unique circle through A and B that is orthogonal to ω (as given by Theorem 55.2). Any M-transformation which maps point A to point B and maps point B to point A must map the circle $\mathscr{C}$ to itself (since two points in Ω uniquely determine a hyperbolic line; if the points are collinear with the center of Ω then the hyperbolic line is a diameter of the unit circle, otherwise it is a segment of a circle that intersects ω at right angles). By Corollary 55.A, by taking $\mathscr{C} = \mathscr{C}_A = \mathscr{C}_B$, we have that there are only two M-transformations which map A to B and $\mathscr{C}$ to itself. The proof of Corollary 55.A shows that one of the M-transformations preserves the orientation on $\mathscr{C}$ and the other reverses the orientation. However, the M-transformation which preserves the orientation on $\mathscr{C}$ cannot map A to B (since the orientation of $\mathscr{C}$ from A to B is opposite that from B to A).

Theorem 55.3 (continued 1)

Theorem 55.3. There exists a unique M-transformation which interchanges two given points A and B of Ω.

Proof (continued). So the only possible M-transformation which interchanges points A and B is the one which reverses the orientation of $\mathscr{C}$. Let M' be the M-transformation which maps A to B and reverses the orientation of $\mathscr{C}$. We next show that M' maps B to A, completing the proof.

Let X_1 and X_2 be the points of intersection of $\mathscr{C}$ with ω. Let the transform of B under M' be A' (we will show that $A' = B$, as desired). Since $\mathscr{C}$ and ω are mapped by M' onto themselves, the set of two intersections X_1 and X_2 of $\mathscr{C}$ and ω is mapped onto itself. Hence either X_1 is mapped to X_1 and X_2 is mapped to X_2, or X_1 is mapped to X_2 and X_2 is mapped to X_1. Since M' reverses the orientation of $\mathscr{C}$, then we cannot have M' fixing X_1 and X_2. So we must have X_1 and X_2 interchanged by M'.

Theorem 55.3 (continued 2)

Theorem 55.3. There exists a unique M-transformation which interchanges two given points A and B of Ω.

Proof (continued). Since the cross-ratio is invariant under a Möbius transformation by Theorem 53.4, then

$$(X_1, X_2; A, B) = (M'(X_1), M'(X_2); M'(A), M'(B)) = (X_2, X_1; B, A').$$

For any cross-ratio we have by Theorem 53.3 that $(X_1, X_2; A, B) = (X_2, X_1; B, A)$, so that we must have $(X_2, X_1; B, A) = (X_2, X_1; B, A')$. So, by the definition of cross-ratio, we have $A = A'$, as desired. □