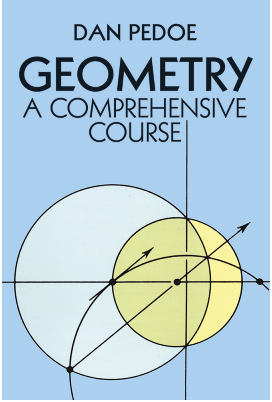
# Real Analysis

**Chapter VI. Mappings of the Inversive Plane**

55. $M$-Transformations—Proofs of Theorems



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# Theorem 55.2

**Theorem 55.2.** Let $X$ and $A$ be two points of $\Omega$ and let $\xi$ and $\alpha$ be directions through $X$ and $A$, respectively. Then there is a unique $M$-transformation which maps $X$ on $A$, and maps the direction $\xi$ onto the direction $\alpha$.

**Proof.** First, by Exercise 55.1 there is exactly one circle orthogonal to $\omega$ that passes through point $X$ and has vector $\xi$ as its direction vector at $X$ (Pedoe describes the direction-condition in terms of tangency to the [Euclidean] straight line determined by vector $\xi$.) Denote this circle as $\mathscr{C}_X$, and let $X_1$ and $X_2$ be the points of intersection of $\mathscr{C}_X$ and $\omega$. Similarly, let $\mathscr{C}_A$ be the unique circle through point $A$ with direction $\alpha$ at $A$ and orthogonal to $\omega$. Let $A_1$ and $A_2$ be the points of intersection of $\mathscr{C}_X$ with $\omega$. Since Möbius transformations preserve angles of intersection by Theorem 54.3, and an $M$-transformation maps $\omega$ onto itself, circle orthogonal to $\omega$ are mapped onto circles orthogonal to $\omega$.

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# Theorem 55.2 (continued 1)

**Proof (continued).** Now any $M$-transformation which maps point $A$ and vector $\xi$ onto point $A$ and vector $\alpha$, must map $\mathscr{C}_X$ onto $\mathscr{C}_A$ since $\mathscr{C}_X$ is mapped onto a circle through $A$ with direction $\alpha$ at $A$ and orthogonal to $\omega$, and there is only one such circle (by Exercise 55.1), namely $\mathscr{C}_A$.

Next label the points of intersection of $\mathscr{C}_X$ and $\mathscr{C}_A$ with $\omega$, such that the direction $\xi$ of $\mathscr{C}_X$ is that of $X_1$ towards $X_2$ and such that the direction $\alpha$ of $\mathscr{C}_A$ is that of $A_1$ towards $A_2$ (direction can be put on a line or circle by giving it parametrically; this is also how tangent vectors such as $\xi$ and $\alpha$ can be determined by taking derivatives with respect to the parameter).

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**Proof (continued).** Now any $M$-transformation which maps point $A$ and vector $\xi$ onto point $A$ and vector $\alpha$, must map $\mathscr{C}_X$ onto $\mathscr{C}_A$ since $\mathscr{C}_X$ is mapped onto a circle through $A$ with direction $\alpha$ at $A$ and orthogonal to $\omega$, and there is only one such circle (by Exercise 55.1), namely $\mathscr{C}_A$.

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# Theorem 55.2 (continued 1)

**Proof (continued).** Now any $M$-transformation which maps point $A$ and vector $\xi$ onto point $A$ and vector $\alpha$, must map $\mathscr{C}_X$ onto $\mathscr{C}_A$ since $\mathscr{C}_X$ is mapped onto a circle through $A$ with direction $\alpha$ at $A$ and orthogonal to $\omega$, and there is only one such circle (by Exercise 55.1), namely $\mathscr{C}_A$.

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# Theorem 55.2 (continued 2)

**Theorem 55.2.** Let $X$ and $A$ be two points of $\Omega$ and let $\xi$ and $\alpha$ be directions through $X$ and $A$, respectively. Then there is a unique $M$-transformation which maps $X$ on $A$, and maps the direction $\xi$ onto the direction $\alpha$.

**Proof (continued).** Now the Möbius transformation maps the circle through $X_1$ and $X_2$ which is orthogonal to $\mathscr{C}_X$ (namely, circle $\omega$) must be mapped to the circle through $A_1$ and $A_2$ which is orthogonal to $\mathscr{C}_A$ (also circle $\omega$; the preservation of orthogonality is given by Theorem 54.3). That is, $\omega$ is mapped to $\omega$ by the Möbius transformation. Since the interior point $X$ of $\Omega$ is mapped to the interior point $A$ of $\Omega$, then the interior of $\omega$ is mapped to the interior of $\omega$; that is, the unique Möbius transformation is an $M$-transformation, as needed. □

# Corollary 55.A

**Corollary 55.A.** Let $\mathscr{C}_A$ be a circle through point $A$ of $\Omega$ to $\omega$, and let $\mathscr{C}_B$ be a circle through a point $B$ of $\Omega$ orthogonal to $\omega$. Then there exist just two $M$-transformations which map $A$ on $B$ and $\mathscr{C}_A$ on $\mathscr{C}_B$.

**Proof.** For a given direction tangent to $\mathscr{C}_A$ at point $A$ and a given tangent to $\mathscr{C}_B$ at point $B$ there is, by Theorem 55.2, a unique $M$-transformation mapping $A$ to $X$ and mapping $\mathscr{C}_A$ to $\mathscr{C}_B$. Since there are two directions tangent to $\mathscr{C}_B$ at $B$ then for a given direction tangent to $\mathscr{C}_A$ at $A$ there are two such $M$-transformations (reversing the direction of the tangent to $\mathscr{C}_A$ at $A$ results in the same two $M$-transformations). Therefore, there are two such $M$-transformations, as claimed. □

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# Theorem 55.3

**Theorem 55.3.** There exists a unique $M$-transformation which interchanges two given points $A$ and $B$ of $\Omega$.

**Proof.** Let $\mathscr{C}$ be the unique circle through $A$ and $B$ that is orthogonal to $\omega$ (as given by Theorem 55.2). Any $M$-transformation which maps point $A$ to point $B$ and maps point $B$ to point $A$ must map the circle $\mathscr{C}$ to itself (since two points in $\Omega$ uniquely determine a hyperbolic line; if the points are collinear with the center of $\Omega$ then the hyperbolic line is a diameter of the unit circle, otherwise it is a segment of a circle that intersects $\omega$ at right angles).

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# Theorem 55.3 (continued 1)

**Theorem 55.3.** There exists a unique $M$-transformation which interchanges two given points $A$ and $B$ of $\Omega$.

**Proof (continued).** So the only possible $M$-transformation which interchanges points $A$ and $B$ is the one which reverses the orientation of $\mathscr{C}$. Let $M'$ be the $M$-transformation which maps $A$ to $B$ and reverses the orientation of $\mathscr{C}$. We next show that $M'$ maps $B$ to $A$, completing the proof.

Let $X_1$ and $X_2$ be the points of intersection of $\mathscr{C}$ with $\omega$. Let the transform of $B$ under $M'$ be $A'$ (we will show that $A' = B$, as desired). Since $\mathscr{C}$ and $\omega$ are mapped by $M'$ onto themselves, the set of two intersections $X_1$ and $X_2$ of $\mathscr{C}$ and $\omega$ is mapped onto itself. Hence either $X_1$ is mapped to $X_1$ and $X_2$ is mapped to $X_2$, or $X_1$ is mapped to $X_2$ and $X_2$ is mapped to $X_1$. Since $M'$ reverses the orientation of $\mathscr{C}$, then we cannot have $M'$ fixing $X_1$ and $X_2$. So we must have $X_1$ and $X_2$ interchanged by $M'$.

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# Theorem 55.3 (continued 2)

**Theorem 55.3.** There exists a unique $M$-transformation which interchanges two given points $A$ and $B$ of $\Omega$.

**Proof (continued).** Since the cross-ratio is invariant under a Möbius transformation by Theorem 53.4, then

$$(X_1, X_2; A, B) = (M'(X_1), M'(X_2); M'(A), M'(B)) = (X_2, X_1; B, A').$$

For any cross-ratio we have by Theorem 53.3 that $(X_1, X_2; A, B) = (X_2, X_1; B, A)$, so that we must have $(X_2, X_1; B, A) = (X_2, X_1; B, A')$. So, by the definition of cross-ratio, we have $A = A'$, as desired. □