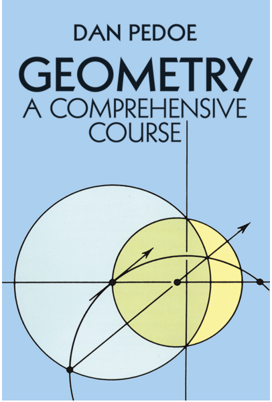
Real Analysis

Chapter VI. Mappings of the Inversive Plane

57. Hyperbolic Triangles and Parallels—Proofs of Theorems



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Theorem 57.1.I. The SAS Theorem. Given two p-triangles ABC and $A'B'C'$ with equal orientation, the congruences $AB \stackrel{p}{=} A'B'$, $AC \stackrel{p}{=} A'C'$, and $\sphericalangle CAB \stackrel{p}{=} C'A'B'$ imply the p-congruence of the triangles, so that $\sphericalangle ABC \stackrel{p}{=} \sphericalangle A'B'C'$, $\sphericalangle ACB \stackrel{p}{=} \sphericalangle A'C'B'$ and $BC \stackrel{p}{=} B'C'$.

Proof. We just need to give an M-transformation which maps A to A', B to B', and C to C'. Since $AB \stackrel{p}{=} A'B'$ by hypothesis, then there is an M-transformation which maps A to A' and maps B to B' (by the definition of congruent segments); denote it as M_1. By Exercise 56.2, this M-transformation M_1 is unique; we now consider the effect of M_1 on p-line AC.

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Proof (continued). By hypothesis $AC \stackrel{p}{=} A'C'$, so there is an M-transformation M_2 which maps A to A' and maps C to C', and by Exercise 56.2 this M-transformation is unique. So M_1 and M_2 both map A to A' and both map the direction of p-line AC to the direction of p-line $A'C'$, and hence by Theorem 55.2 we must have $M_1 = M_2$. But M_1 maps B to B', so that M_2 does as well. Since M_2 maps C to C' also, then $BC \stackrel{p}{=} B'C'$, as claimed. By Theorem 54.3, angles ABC and $A'B'C'$ are equal and angles ACB and $A'C'B'$ are equal; that is, $\measuredangle ABC \stackrel{p}{=} \measuredangle A'B'C'$ and $\measuredangle ACB \stackrel{p}{=} \measuredangle A'C'B'$, as claimed. □

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