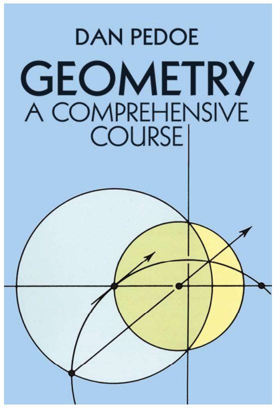
# Real Analysis

**Chapter VI. Mappings of the Inversive Plane**

58. More Hyperbolic Triangles and Distance—Proofs of Theorems



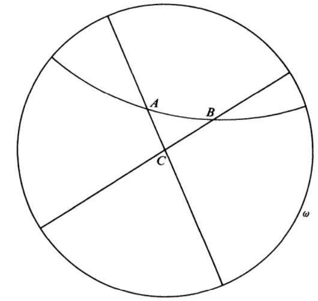





## Theorem 58.1

**Theorem 58.1.** The sum of the angles of a $p$-triangle $ABC$ is (strictly) less than $\pi$.

**Proof.** Let $C$ be the center of $\omega$ (see Figure 58.1). The the $p$-lines $CA$ and $CB$ are diameters of $\omega$ and the $p$-line $AB$ is an arc of a circle which is orthogonal to $\omega$. If two circles are orthogonal, then their centers do not both lie inside one of the circles (we accept this based on our geometric intuition). So the center of the $p$-line $AB$ lies outside $\omega$, and the center $C$ of circle $\omega$ lies outside the circle which determines the $p$-line $AB$. Hence the $p$-line $AB$ is convex to $C$, as shown in Figure 58.1. This implies the angles of the curvilinear triangle $ABC$ at $A$ and at $B$ are less than the angles of the Euclidean triangle $ABC$ at $A$ and at $B$. Since the angle-sum of the Euclidean triangle$ABC$ is equal to $\pi$, the angle-sum of the $p$-triangle $ABC$ is less than $\pi$.

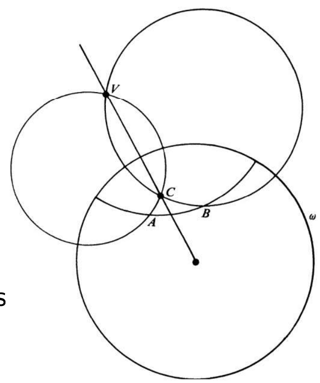







## Theorem 58.1 (continued)

**Proof (continued).** Now suppose no vertex of the $p$-triangle $ABC$ is the center of $\omega$. Let $V$ be the other point of intersection of the circles which determine the $p$-lines $AC$ and $BC$. Then $V$ and $C$ are inverse points in $\omega$ (see "Section 20. Inversion"). Let $\mathscr{C}$ be the circle which has $V$ as its center and which is orthogonal to $\omega$.




Inverting in $\mathscr{C}$ maps $\omega$ to itself and maps circles through $V$ to lines. Since $C$ is the intersection of two circles orthogonal to $\omega$ then $C$ inverts to the center of $\omega$, and the circles through $V$ invert into diameters of $\omega$. So the $p$-triangle $ABC$ inverts onto $p$-triangle $A'B'C'$, where $C'$ is the center of $\omega$. As shown above, this implies that the angle-sum of $A'B'C'$ is less than $\pi$. Since inversion preserves the measure of angles (but not the sense) then the angle sum of $ABC$ is less than $\pi$, as claimed. □





## Theorem 58.2

**Theorem 58.2.** In any $p$-triangle, if one of the $p$-sides be extended, the exterior angle is greater than either of the interior and opposite angles.

**Proof.** The proof is similar to the second part of Theorem 58.1. Consider the $p$-line $CB$ extended to produce the exterior angle at $B$ of $p$-triangle $ABC$. With $C'$ as the center of $\omega$, inversion with respect to the circle $\mathscr{C}$ introduced in the proof of Theorem 58.1, inversion with respect to $\mathscr{C}$ maps $C$ to $C'$. The $p$-line $B'A'$ is convex to point $C'$, so the exterior angle at $B'$ of $p$-triangle $A'B'C'$ is greater than the same angle for the Euclidean triangle $A'B'C'$, and the interior angle at $A'$ of $p$-triangle $A'B'C'$ is less than the same angle for the Euclidean triangle $A'B'C'$.

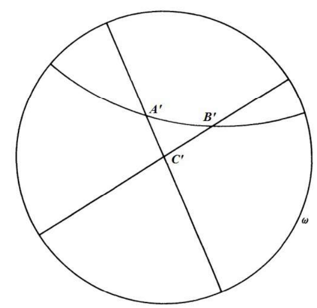



Figure 58.1 (revised)

## Theorem 58.2 (continued)

**Theorem 58.2.** In any $p$-triangle, if one of the $p$-sides be extended, the exterior angle is greater than either of the interior and opposite angles.

**Proof (continued).** By Euclid's Book I Proposition 32, the exterior angle at $B'$ for the Euclidean triangle $A'B'C'$ is greater than the angle interior angle at $A'$. So in the $p$-triangle $A'B'C'$, the exterior angle at $B'$ is "even more" greater than the interior angle at $A'$. Since the angles of $p$-triangle $ABC$ are unchanged in measure by an inversion (though not in sense), then the exterior angle at $B$ of $p$-triangle $ABC$ exceeds interior angle $A$, as claimed. □





## Theorem 58.A

**Theorem 58.A.** If the points $\alpha$, $A$, B, $\beta$ are in this order on a $p$-line, where $\alpha$ and $\beta$ are on $\omega$, then the cross-ratio $(\alpha, \beta; A, B)$ is a positive real number less than 1.

**Proof.** There is a unique Möbius transformation which maps points $\alpha$, $A$, and $\beta$ onto three given points $\alpha'$, $A'$, and $\beta'$ of the real line (where we choose $A'$ to lie between $\alpha'$ and $\beta'$) by Theorem 53.2. For any pair of points $A$, $B$ on the $p$-line whose terminal points on $\omega$ are $\alpha$ and $\beta$, the points $A'$, $B'$ will like on the segment of the real line with endpoints $\alpha'\beta'$. Since a Möbius transformation preserves cross-ratios (Theorem 53.4) then $(\alpha, \beta; A, B) = (\alpha', \beta'; A', B')$, which is real by Theorem 53.5. Now the order $\alpha, A, B, \beta$ on the $p$-line is the same as the order $\alpha', A', B', \beta'$ on the real line. The cross-ratio (by definition) is

$$(\alpha', \beta'; A', B') = \left(\frac{\alpha' - A'}{\beta' - A'}\right) \Big/ \left(\frac{\alpha' - B'}{\beta' - B'}\right) = \left(\frac{\alpha' - A'}{A' - \beta'}\right) \Big/ \left(\frac{\alpha' - B'}{B' - \beta'}\right).$$





## Theorem 58.A (continued)

**Proof (continued).** ...

$$(\alpha', \beta'; A', B') = \left(\frac{\alpha' - A'}{A' - \beta'}\right) \Big/ \left(\frac{\alpha' - B'}{B' - \beta'}\right).$$

Since $A'$ is between $\alpha'$ and $\beta'$ on the real line, then $(\alpha' - A')/(A' - \beta')$ is positive. Similarly, since $B'$ is between $A'$ and $\beta'$ on the real line, then $(\alpha' - B')/(B' - \beta')$ is positive and so the cross-ratio is positive. In addition, since $B'$ is between $A'$ and $\beta'$ on the real line, then $A'$ is closer to $\alpha'$ than $B'$ is; that is, $|\alpha' - A'| < |\alpha' - B'|$. Also $B'$ is closer to $\beta'$ than $A'$ is; that is $|B' - \beta'| < |A' - \beta'|$ or $|A' - \beta'| > |B' - \beta'|$. Therefore,

$$\left(\frac{\alpha' - A'}{A' - \beta'}\right) = \left|\frac{\alpha' - A'}{A' - \beta'}\right| < \left|\frac{\alpha' - B'}{B' - \beta'}\right| = \left(\frac{\alpha' - B'}{B' - \beta'}\right),$$

and the cross-ratio $(\alpha', \beta'; A', B')$ is less than 1. So $(\alpha, \beta; A, B) = (\alpha', \beta'; A', B')$ is less than 1, as claimed. □





## Theorem 58.2.I

**Theorem 58.2.I.** If $A$, $B$, and $C$ are $p$-points, in this order, on a $p$-line then $d(A, C) = d(A, B) + d(B, C)$.

**Proof.** Suppose that $\alpha, A, B, C, \beta$ lie, in this order, on a $p$-line, where $\alpha$ and $\beta$ are the intersections of the $p$-line with $\omega$. Then the cross-ratios $(\alpha, \beta; A, B)$, $(\alpha, \beta; B, C)$, and $(\alpha, \beta; A, C)$ are positive real numbers less than 1 by Theorem 58.A. By Exercise 53.5, we have

$$(\alpha, \beta; A, C) = (\alpha, \beta; A, B)\,(\alpha, \beta; B, C),$$

$$\text{or } \log(\alpha, \beta; A, C) = \log(\alpha, \beta; A, B) + \log(\alpha, \beta; B, C),$$

and each of these three logarithms are negative numbers, as shown in the proof of Theorem 58.A so that

$$|\log(\alpha, \beta; A, C)| = |\log(\alpha, \beta; A, B)| + |\log(\alpha, \beta; B, C)|.$$

Therefore, $d(A, C) = d(A, B) + d(B, C)$, as claimed. □

## Theorem 58.B

**Theorem 58.B.** The distance function $d$ is invariant under $M^*$-transformations. That is, for $A, B \in \Omega$ and $M$ and $M^*$-transformation, $d(A, B) = d(M(A), M(B))$.

**Proof.** By Theorem 53.4, an $M_+$-transformation leaves cross-ratios unchanged, and by Exercise 57.7 an $M_-$-transformation maps a cross-ratio to its conjugate complex value. Let $A, B \in \Omega$ and let $\alpha, \beta$ be the points of intersection of the $p$-line $AB$ with $\omega$. So $\alpha, \beta, A, B$ lie on the same $p$-line so that the cross-ratio $(\alpha, \beta; A, B)$ is real by Theorem 53.5. So both an $M_+$-transformation and an $M_-$-transformation leave $(\alpha, \beta; A, B)$ invariant and hence map this $p$-line to another $p$-line. Now an $M^*$-transformation maps $\omega$ to itself so the images of $\alpha$ and $\beta$ under such a transformation are also on $\omega$. So with the images of $\alpha, \beta, A, B$ as $\alpha', \beta', A', B'$ we have $(\alpha, \beta; A, B) = (\alpha', \beta'; A', B')$, or $d(A, B) = d(A', B')$, as claimed. □





## Theorem 58.C

**Theorem 58.C.** If for points $A, B \in \Omega$ we have $d(A, B) = d(A', B')$, then there is an $M_+$-transformation which maps $A$ to $A'$ and maps $B$ to $B'$.

**Proof.** Let $A, B \in \Omega$ so that $AB$ is the $p$-line containing $A$ and $B$, and let this $p$-line intersect $\omega$ at $\alpha$ and $\beta$ where the order of the points on the $p$-line are $\alpha, A, B, \beta$. Similarly, let $A', B' \in \Omega$ so that $A'B'$ is the $p$-line containing $A'$ and $B'$, and let this $p$-line intersect $\omega$ at $\alpha'$ and $\beta'$ where the order of the points on the $p$-line are $\alpha', A', B', \beta'$. By Theorem 55.2, there is a unique $M_+$-transformation, $M$, which maps $\alpha$ to $\alpha'$, $A$ to $A'$, and $\beta$ to $\beta'$. We now show that this $M_+$-transformation also maps $B$ to $B'$. Since $d(A, B) = d(A', B')$ by hypothesis, then $|\log(\alpha, \beta; A, B)| = |\log(\alpha', \beta'; A', B')|$. By Theorem 58.A each cross-ratio is a positive real number less than 1. So both logarithms are negative real numbers and equal in magnitude, and hence are equal. That is, $(\alpha, \beta; A, B) = (\alpha', \beta'; A', B')$.





## Theorem 58.C (continued)

**Theorem 58.C.** If for points $A, B \in \Omega$ we have $d(A, B) = d(A', B')$, then there is an $M_+$-transformation which maps $A$ to $A'$ and maps $B$ to $B'$.

**Proof (continued).** Applying $M$ and using the invariance of the cross-ratio (Theorem 53.4), we have

$$(\alpha, \beta; A, B) = (M(\alpha), M(\beta); M(A), M(B)) = (\alpha', \beta'; A', M(B))$$
$$= (\alpha', \beta'; A', B'),$$

so that we must have $M(B) = B'$ (as can be seen by expanding the the cross-ratios). So $M$ is the desired $M_+$-transformation. □





## Theorem 58.2.II

**Theorem 58.2.II.** In a $p$-triangle $ABC$, if $d(A, B) = d(A, C)$, then the $p$-angles $ABC$ and $ACB$ are congruent.

**Proof.** Consider the two $p$-triangles $ABC$ and $ACB$. Then we have $AB \overset{p}{=} AC$ and $AC \overset{p}{=} AB$ by hypothesis, and $\sphericalangle BAC \overset{p}{=} \sphericalangle CAB$. So by Theorem 57.1 (the General SAS Theorem), we have that the $p$-triangles $ABC$ and $ACB$ are $p$-congruent. Therefore $\sphericalangle ABC \overset{p}{=} \sphericalangle ACB$, as claimed. □

# Theorem 58.2.III

**Theorem 58.2.III.** In any $p$-triangle $ABC$, the greatest side is opposite the greatest angle.

**Proof.** Suppose that $d(A,C) > d(A,B)$, as shown in Figure 58.6. Mark off a $p$-segment $AB'$ on $p$-line $AC$ equal to $AB$ so that $d(A,B') = d(A,B)$ (which can be done by Note 58.B). Then $B'$ lies on a $p$-segment $AC$ and the $p$-line $BB'$ lies inside the angle $ABC$. Since $d(A,B') = d(A,B)$, then by Theorem 58.2.II then angles $ABB'$ and $AB'B$ are equal.

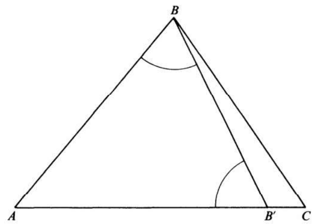


Figure 58.6

Since angle $ABC$ is greater than angle $ABB'$ by construction, then angle $ABC$ is greater then angle $AB'B$. But angle $AB'B$ is the exterior angle to angle $BB'C$ in $p$-triangle $BB'C$, and therefore exceeds the interior angle $B'CB$ by Theorem 58.2. But angle $B'CB$ is angle $ACB$, so angle $ABC$ is greater than angle $ABB'$ which is greater than angle $B'CB$ which equals angle $ACB$.





# Theorem 58.2.III (continued)

**Theorem 58.2.III.** In any $p$-triangle $ABC$, the greatest side is opposite the greatest angle.

**Proof.** That is, angle $ABC$ is greater than angle $ACB$. We arrive at this conclusion under the supposition that $d(A,C) > d(A,B)$. So we have that larger angles are opposite larger sides in a $p$-triangle, and hence the greatest side is opposite the greatest angle, as claimed. □

**Note.** We have also shown that the least side is opposite the least angle.





# Theorem 58.2.IV

**Theorem 58.2.IV.** In any $p$-triangle $ABC$, $d(A,B) + d(B,C) > d(A,C)$.

**Proof.** Notice that if either $AB$ or $BC$ is the longest side of the $p$-triangle, then the inequality holds. So we can assume without loss of generality that the longest side is side $AC$. Create angle $B'AC$ equal to angle $BAC$ where point $B'$ is on the side of $p$-line $AC$ opposite of the side of $AC$ On the opposite side of the $p$-line $AC$ to $B$, mark off angle $B''AC$ equal which is equal to angle $BAC$ (which can be accomplished by a Möbius transformation that consists of a rotation about point $A$ that sends $p$-line $AB$ to $p$-line $AC$). Choose point $B'$ on $p$-line $AB''$ where we $B'$ is on the same side of $A$ on the $p$-line as is $B''$ and $d(A,B') = d(A,B)$ (which can be done by Note 58.B). Then angle $B'AC$ equals angle $BAC$ (in size, but maybe not in sense); see Figure 58.7.

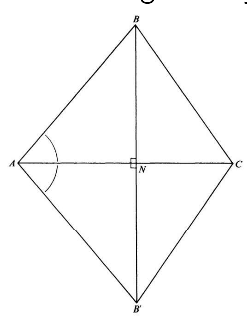


Figure 58.7





# Theorem 58.2.IV (continued)

**Theorem 58.2.IV.** In any $p$-triangle $ABC$, $d(A,B) + d(B,C) > d(A,C)$.

**Proof (continued).** Let the point of intersection of $p$-lines $BB'$ and $AC$ be $N$ (since $B$ and $B'$ are on opposite sides of $AC$ then there must be some such point $N$). Then $p$-triangles $ANB$ and $ANB'$ are $p$-congruent by by Side-Angle-Side (Theorem 57.1.II). Since the sum of the angles $BNA$ and $B'NA$ is two right angles, so (since angles $ANB$ equals angle $ANB'$) are both right angles. Then angles $BNC$ and $B'NC$ are also right angles. So by Theorem 58.2.III, we have $d(A,B) > d(B,C)$ and $d(B,C) > d(N,C)$. Hence, if $N$ lies on the $p$-segment $AC$, then we have by Theorem 58.2.I that $d(A,B) + d(B,C) > d(A,N) + d(N,C) = d(A,C)$, as claimed.

If $N$ is not on the $p$-segment $AC$, we still have $d(A,B) > d(A,N)$ and since in this case $d(A,N) > d(A,C)$, we would have $d(A,B) > d(A,C)$, but this contradicts the fact that we are assuming that $AC$ is the longest side of $p$-triangle $ABC$. So we cannot have $N$ not on $p$-segment $AC$, so that the result holds as aregued above. □