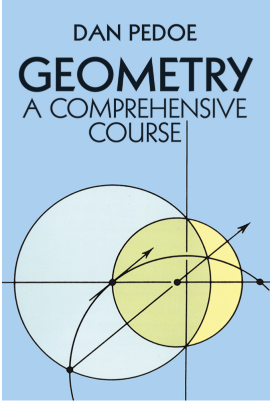
Real Analysis

Chapter VI. Mappings of the Inversive Plane

58. More Hyperbolic Triangles and Distance—Proofs of Theorems



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Theorem 58.1. The sum of the angles of a p-triangle ABC is (strictly) less than π.

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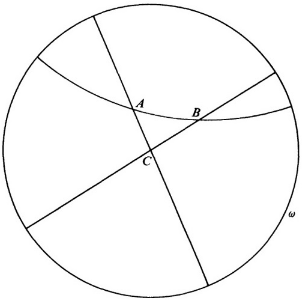

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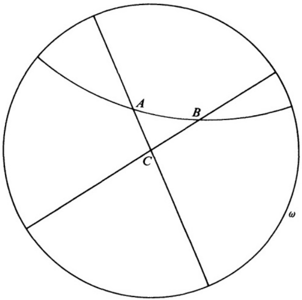
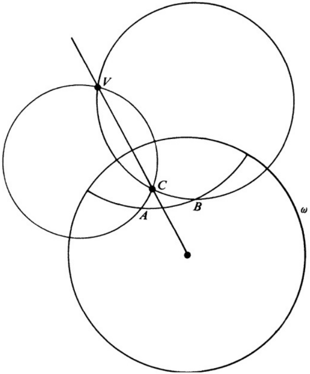
Proof. Let C be the center of ω (see Figure 58.1). The the p-lines CA and CB are diameters of ω and the p-line AB is an arc of a circle which is orthogonal to ω. If two circles are orthogonal, then their centers do not both lie inside one of the circles (we accept this based on our geometric intuition). So the center of the p-line AB lies outside ω, and the center C of circle ω lies outside the circle which determines the p-line AB. Hence the p-line AB is convex to C, as shown in Figure 58.1. This implies the angles of the curvilinear triangle ABC at A and at B are less than the angles of the Euclidean triangle ABC at A and at B. Since the angle-sum of the Euclidean triangleABC is equal to π, the angle-sum of the p-triangle ABC is less than π.



Theorem 58.1 (continued)

Proof (continued). Now suppose no vertex of the p-triangle ABC is the center of ω. Let V be the other point of intersection of the circles which determine the p-lines AC and BC. Then V and C are inverse points in ω (see "Section 20. Inversion"). Let $\mathscr{C}$ be the circle which has V as its center and which is orthogonal to ω.



Inverting in $\mathscr{C}$ maps ω to itself and maps circles through V to lines. Since C is the intersection of two circles orthogonal to ω then C inverts to the center of ω, and the circles through V invert into diameters of ω. So the p-triangle ABC inverts onto p-triangle $A'B'C'$, where C' is the center of ω. As shown above, this implies that the angle-sum of $A'B'C'$ is less than π. Since inversion preserves the measure of angles (but not the sense) then the angle sum of ABC is less than π, as claimed. □

Theorem 58.1 (continued)

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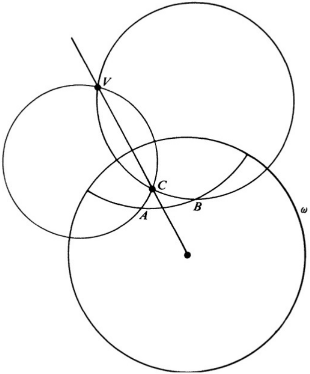

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Theorem 58.2

Theorem 58.2. In any p-triangle, if one of the p-sides be extended, the exterior angle is greater than either of the interior and opposite angles.

Proof. The proof is similar to the second part of Theorem 58.1. Consider the p-line CB extended to produce the exterior angle at B of p-triangle ABC. With C' as the center of ω, inversion with respect to the circle $\mathscr{C}$ introduced in the proof of Theorem 58.1, inversion with respect to $\mathscr{C}$ maps C to C'.

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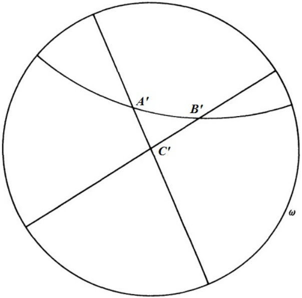

Figure 58.1 (revised)

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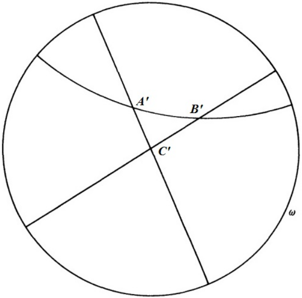

Figure 58.1 (revised)

Theorem 58.2 (continued)

Theorem 58.2. In any p-triangle, if one of the p-sides be extended, the exterior angle is greater than either of the interior and opposite angles.

Proof (continued). By Euclid's Book I Proposition 32, the exterior angle at B' for the Euclidean triangle $A'B'C'$ is greater than the angle interior angle at A'. So in the p-triangle $A'B'C'$, the exterior angle at B' is "even more" greater than the interior angle at A'. Since the angles of p-triangle ABC are unchanged in measure by an inversion (though not in sense), then the exterior angle at B of p-triangle ABC exceeds interior angle A, as claimed. □

Theorem 58.A

Theorem 58.A. If the points α, A, B, β are in this order on a *p*-line, where α and β are on ω, then the cross-ratio $(\alpha, \beta; A, B)$ is a positive real number less than 1.

Proof. There is a unique Möbius transformation which maps points α, A, and β onto three given points α', A', and β' of the real line (where we choose A' to lie between α' and β') by Theorem 53.2. For any pair of points A, B on the *p*-line whose terminal points on ω are α and β, the points A', B' will like on the segment of the real line with endpoints $\alpha'\beta'$.

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$$(\alpha', \beta'; A', B') = \left(\frac{\alpha' - A'}{\beta' - A'}\right) \Big/ \left(\frac{\alpha' - B'}{\beta' - B'}\right) = \left(\frac{\alpha' - A'}{A' - \beta'}\right) \Big/ \left(\frac{\alpha' - B'}{B' - \beta'}\right).$$

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Theorem 58.A (continued)

Proof (continued). ...

$$(\alpha', \beta'; A', B') = \left(\frac{\alpha' - A'}{A' - \beta'}\right) \Big/ \left(\frac{\alpha' - B'}{B' - \beta'}\right).$$

Since A' is between α' and β' on the real line, then $(\alpha' - A')/(A' - \beta')$ is positive. Similarly, since B' is between A' and β' on the real line, then $(\alpha' - B')/(B' - \beta')$ is positive and so the cross-ratio is positive. In addition, since B' is between A' and β' on the real line, then A' is closer to α' than B' is; that is, $|\alpha' - A'| < |\alpha' - B'|$. Also B' is closer to β' than A' is; that is $|B' - \beta'| < |A' - \beta'|$ or $|A' - \beta'| > |B' - \beta'|$. Therefore,

$$\left(\frac{\alpha' - A'}{A' - \beta'}\right) = \left|\frac{\alpha' - A'}{A' - \beta'}\right| < \left|\frac{\alpha' - B'}{B' - \beta'}\right| = \left(\frac{\alpha' - B'}{B' - \beta'}\right),$$

and the cross-ratio $(\alpha', \beta'; A', B')$ is less than 1. So $(\alpha, \beta; A, B) = (\alpha', \beta'; A', B')$ is less than 1, as claimed. □

Theorem 58.A (continued)

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and the cross-ratio $(\alpha', \beta'; A', B')$ is less than 1. So $(\alpha, \beta; A, B) = (\alpha', \beta'; A', B')$ is less than 1, as claimed. □

Theorem 58.2.I

Theorem 58.2.I. If A, B, and C are p-points, in this order, on a p-line then $d(A, C) = d(A, B) + d(B, C)$.

Proof. Suppose that α, A, B, C, β lie, in this order, on a p-line, where α and β are the intersections of the p-line with ω. Then the cross-ratios $(\alpha, \beta; A, B)$, $(\alpha, \beta; B, C)$, and $(\alpha, \beta; A, C)$ are positive real numbers less than 1 by Theorem 58.A.

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$$(\alpha, \beta; A, C) = (\alpha, \beta; A, B)\,(\alpha, \beta; B, C),$$

$$\text{or } \log(\alpha, \beta; A, C) = \log(\alpha, \beta; A, B) + \log(\alpha, \beta; B, C),$$

and each of these three logarithms are negative numbers, as shown in the proof of Theorem 58.A so that

$$|\log(\alpha, \beta; A, C)| = |\log(\alpha, \beta; A, B)| + |\log(\alpha, \beta; B, C)|.$$

Therefore, $d(A, C) = d(A, B) + d(B, C)$, as claimed. □

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Therefore, $d(A, C) = d(A, B) + d(B, C)$, as claimed. □

Theorem 58.B

Theorem 58.B. The distance function d is invariant under M^*-transformations. That is, for $A, B \in \Omega$ and M and M^*-transformation, $d(A, B) = d(M(A), M(B))$.

Proof. By Theorem 53.4, an M_+-transformation leaves cross-ratios unchanged, and by Exercise 57.7 an M_--transformation maps a cross-ratio to its conjugate complex value. Let $A, B \in \Omega$ and let α, β be the points of intersection of the p-line AB with ω. So α, β, A, B lie on the same p-line so that the cross-ratio $(\alpha, \beta; A, B)$ is real by Theorem 53.5.

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Theorem 58.C

Theorem 58.C. If for points $A, B \in \Omega$ we have $d(A, B) = d(A', B')$, then there is an M_+-transformation which maps A to A' and maps B to B'.

Proof. Let $A, B \in \Omega$ so that AB is the p-line containing A and B, and let this p-line intersect ω at α and β where the order of the points on the p-line are α, A, B, β. Similarly, let $A', B' \in \Omega$ so that $A'B'$ is the p-line containing A' and B', and let this p-line intersect ω at α' and β' where the order of the points on the p-line are α', A', B', β'. By Theorem 55.2, there is a unique M_+-transformation, M, which maps α to α', A to A', and β to β'. We now show that this M_+-transformation also maps B to B'.

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Proof. Let $A, B \in \Omega$ so that AB is the p-line containing A and B, and let this p-line intersect ω at α and β where the order of the points on the p-line are α, A, B, β. Similarly, let $A', B' \in \Omega$ so that $A'B'$ is the p-line containing A' and B', and let this p-line intersect ω at α' and β' where the order of the points on the p-line are α', A', B', β'. By Theorem 55.2, there is a unique M_+-transformation, M, which maps α to α', A to A', and β to β'. We now show that this M_+-transformation also maps B to B'. Since $d(A, B) = d(A', B')$ by hypothesis, then $|\log(\alpha, \beta; A, B)| = |\log(\alpha', \beta'; A', B')|$. By Theorem 58.A each cross-ratio is a positive real number less than 1. So both logarithms are negative real numbers and equal in magnitude, and hence are equal. That is, $(\alpha, \beta; A, B) = (\alpha', \beta'; A', B')$.

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Theorem 58.C (continued)

Theorem 58.C. If for points $A, B \in \Omega$ we have $d(A, B) = d(A', B')$, then there is an M_+-transformation which maps A to A' and maps B to B'.

Proof (continued). Applying M and using the invariance of the cross-ratio (Theorem 53.4), we have

$$(\alpha, \beta; A, B) = (M(\alpha), M(\beta); M(A), M(B)) = (\alpha', \beta'; A', M(B))$$

$$= (\alpha', \beta'; A', B'),$$

so that we must have $M(B) = B'$ (as can be seen by expanding the the cross-ratios). So M is the desired M_+-transformation. □

Theorem 58.2.II

Theorem 58.2.II. In a p-triangle ABC, if $d(A, B) = d(A, C)$, then the p-angles ABC and ACB are congruent.

Proof. Consider the two p-triangles ABC and ACB. Then we have $AB \stackrel{p}{=} AC$ and $AC \stackrel{p}{=} AB$ by hypothesis, and $\sphericalangle BAC \stackrel{p}{=} \sphericalangle CAB$. So by Theorem 57.1 (the General SAS Theorem), we have that the p-triangles ABC and ACB are p-congruent. Therefore $\sphericalangle ABC \stackrel{p}{=} \sphericalangle ACB$, as claimed. □

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Theorem 58.2.III

Theorem 58.2.III. In any p-triangle ABC, the greatest side is opposite the greatest angle.

Proof. Suppose that $d(A, C) > d(A, B)$, as shown in Figure 58.6. Mark off a p-segment AB' on p-line AC equal to AB so that $d(A, B') = d(A, B)$ (which can be done by Note 58.B). Then B' lies on a p-segment AC and the p-line BB' lies inside the angle ABC. Since $d(A, B') = d(A, B)$, then by Theorem 58.2.II then angles ABB' and $AB'B$ are equal.

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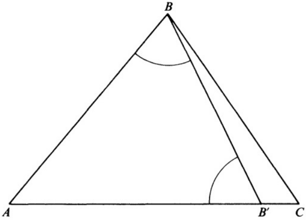

Figure 58.6

Since angle ABC is greater than angle ABB' by construction, then angle ABC is greater then angle $AB'B$. But angle $AB'B$ is the exterior angle to angle $BB'C$ in p-triangle $BB'C$, and therefore exceeds the interior angle $B'CB$ by Theorem 58.2. But angle $B'CB$ is angle ACB, so angle ABC is greater than angle ABB' which is greater than angle $B'CB$ which equals angle ACB.

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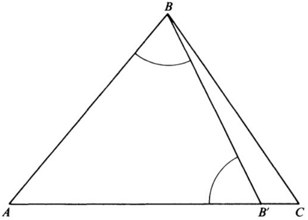

Figure 58.6

Since angle ABC is greater than angle ABB' by construction, then angle ABC is greater then angle $AB'B$. But angle $AB'B$ is the exterior angle to angle $BB'C$ in p-triangle $BB'C$, and therefore exceeds the interior angle $B'CB$ by Theorem 58.2. But angle $B'CB$ is angle ACB, so angle ABC is greater than angle ABB' which is greater than angle $B'CB$ which equals angle ACB.

Theorem 58.2.III (continued)

Theorem 58.2.III. In any p-triangle ABC, the greatest side is opposite the greatest angle.

Proof. That is, angle ABC is greater than angle ACB. We arrive at this conclusion under the supposition that $d(A, C) > d(A, B)$. So we have that larger angles are opposite larger sides in a p-triangle, and hence the greatest side is opposite the greatest angle, as claimed. □

Note. We have also shown that the least side is opposite the least angle.

Theorem 58.2.III (continued)

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Note. We have also shown that the least side is opposite the least angle.

Theorem 58.2.IV

Theorem 58.2.IV. In any p-triangle ABC, $d(A, B) + d(B, C) > d(A, C)$.

Proof. Notice that if either AB or BC is the longest side of the p-triangle, then the inequality holds. So we can assume without loss of generality that the longest side is side AC. Create angle $B'AC$ equal to angle BAC where point B' is on the side of p-line AC opposite of the side of AC On the opposite side of the p-line AC to B, mark off angle $B''AC$ equal which is equal to angle BAC (which can be accomplished by a Möbius transformation that consists of a rotation about point A that sends p-line AB to p-line AC).

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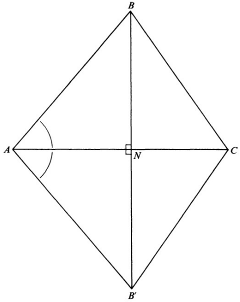

Figure 58.7

Theorem 58.2.IV

Theorem 58.2.IV. In any p-triangle ABC, $d(A,B) + d(B,C) > d(A,C)$.

Proof. Notice that if either AB or BC is the longest side of the p-triangle, then the inequality holds. So we can assume without loss of generality that the longest side is side AC. Create angle $B'AC$ equal to angle BAC where point B' is on the side of p-line AC opposite of the side of AC On the opposite side of the p-line AC to B, mark off angle $B''AC$ equal which is equal to angle BAC (which can be accomplished by a Möbius transformation that consists of a rotation about point A that sends p-line AB to p-line AC). Choose point B' on p-line AB'' where we B' is on the same side of A on the p-line as is B'' and $d(A,B') = d(A,B)$ (which can be done by Note 58.B). Then angle $B'AC$ equals angle BAC (in size, but maybe not in sense); see Figure 58.7.

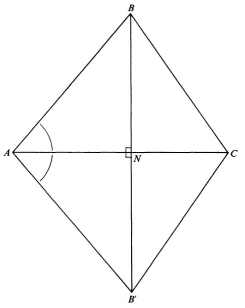

Figure 58.7

Theorem 58.2.IV (continued)

Theorem 58.2.IV. In any p-triangle ABC, $d(A,B) + d(B,C) > d(A,C)$.

Proof (continued). Let the point of intersection of p-lines BB' and AC be N (since B and B' are on opposite sides of AC then there must be some such point N). Then p-triangles ANB and ANB' are p-congruent by by Side-Angle-Side (Theorem 57.1.II). Since the sum of the angles BNA and $B'NA$ is two right angles, so (since angles ANB equals angle ANB') are both right angles. Then angles BNC and $B'NC$ are also right angles. So by Theorem 58.2.III, we have $d(A,B) > d(B,C)$ and $d(B,C) > d(N,C)$. Hence, if N lies on the p-segment AC, then we have by Theorem 58.2.I that $d(A,B) + d(B,C) > d(A,N) + d(N,C) = d(A,C)$, as claimed.

Theorem 58.2.IV (continued)

Theorem 58.2.IV. In any p-triangle ABC, $d(A,B) + d(B,C) > d(A,C)$.

Proof (continued). Let the point of intersection of p-lines BB' and AC be N (since B and B' are on opposite sides of AC then there must be some such point N). Then p-triangles ANB and ANB' are p-congruent by by Side-Angle-Side (Theorem 57.1.II). Since the sum of the angles BNA and $B'NA$ is two right angles, so (since angles ANB equals angle ANB') are both right angles. Then angles BNC and $B'NC$ are also right angles. So by Theorem 58.2.III, we have $d(A,B) > d(B,C)$ and $d(B,C) > d(N,C)$. Hence, if N lies on the p-segment AC, then we have by Theorem 58.2.I that $d(A,B) + d(B,C) > d(A,N) + d(N,C) = d(A,C)$, as claimed.

If N is not on the p-segment AC, we still have $d(A,B) > d(A,N)$ and since in this case $d(A,N) > d(A,C)$, we would have $d(A,B) > d(A,C)$, but this contradicts the fact that we are assuming that AC is the longest side of p-triangle ABC. So we cannot have N not on p-segment AC, so that the result holds as aregued above. □

Theorem 58.2.IV (continued)

Theorem 58.2.IV. In any p-triangle ABC, $d(A, B) + d(B, C) > d(A, C)$.

Proof (continued). Let the point of intersection of p-lines BB' and AC be N (since B and B' are on opposite sides of AC then there must be some such point N). Then p-triangles ANB and ANB' are p-congruent by by Side-Angle-Side (Theorem 57.1.II). Since the sum of the angles BNA and $B'NA$ is two right angles, so (since angles ANB equals angle ANB') are both right angles. Then angles BNC and $B'NC$ are also right angles. So by Theorem 58.2.III, we have $d(A, B) > d(B, C)$ and $d(B, C) > d(N, C)$. Hence, if N lies on the p-segment AC, then we have by Theorem 58.2.I that $d(A, B) + d(B, C) > d(A, N) + d(N, C) = d(A, C)$, as claimed.

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