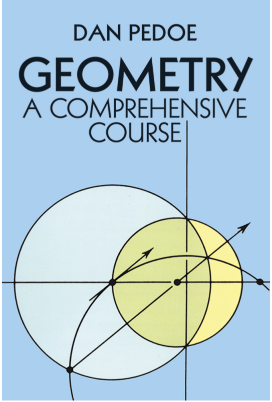
Real Analysis

Chapter VI. Mappings of the Inversive Plane

59. Horocycles—Proofs of Theorems



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Theorem 59.1

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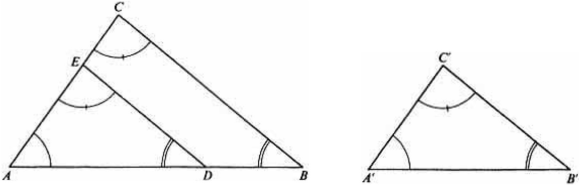

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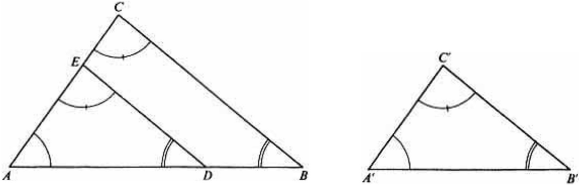


Figure 59.1

Theorem 59.1 (continued 1)

Proof (continued).

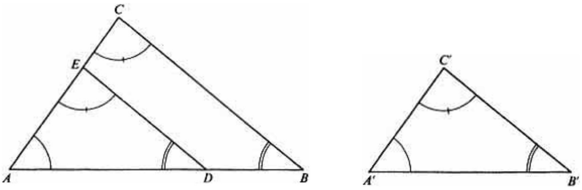
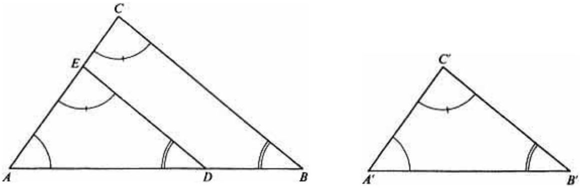


Figure 59.1

Then D lies in the segment AB since $d(A,B) > d(A',B')$ by hypothesis. But point E may coincide with point C, or it may lie in AC extended, or it may lie in segment AC. If E coincides with C, then by Side-Angle-Side (Theorem 57.1) the triangles ADC and $A'B'C'$ are congruent. But then angle ADC = angle $A'B'$ = angle ABC, in CONTRADICTION to Theorem 58.2 which implies that angle $ADC >$ angle ABC. Hence E cannot coincide with C.

Theorem 59.1 (continued 1)

Proof (continued).



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Theorem 59.1 (continued 2)

Proof (continued).

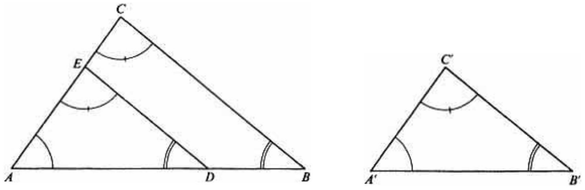

Figure 59.1

If E lies in AC extended then again triangles ADE and $A'B'C'$ are congruent by SAS (Theorem 57.1), and angle $AED =$ angle $A'C'B' =$ angle ACB. Again, this CONTRADICTS Theorem 58.2 which implies angle $ACB >$ angle AED. Hence E cannot lie in AC extended. Therefore E must lie between A and C, as shown in Figure 59.1.

Theorem 59.1 (continued 3)

Proof (continued).

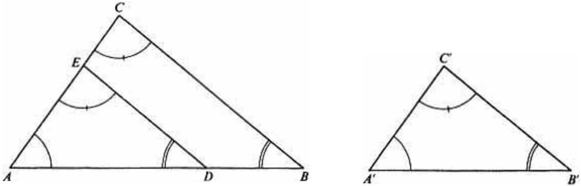

Figure 59.1

But if we consider the quadrilateral $BCED$, the sum of the angles is 2π (because supplemental angles in the quadrilateral). But a quadrilateral can be divided into two triangles, so the Theorem 58.1 the angle sum of a quadrilateral must be less than 2π, another CONTRADICTION.

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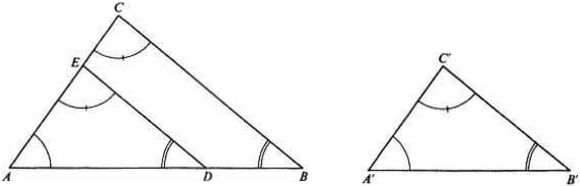

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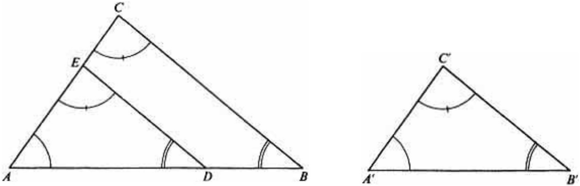

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Theorem 59.2.I

Theorem 59.2.I. A p-circle, center A is a Euclidean circle orthogonal to the family of p-lines which pass through A.

Proof. The p-lines through A all pass through the fixed point A', where A' is the inverse of A in ω by Theorem 20.2 in Chapter II, "Circles." Let P be a point on the p-circle centered at A. Let $\mathscr{C}$ be the p-line through P and A. Then $\mathscr{C}$ passes through A', since inversion with respect to ω interchanges A and A' and maps $\mathscr{C}$ to itself. With α and β as the points of intersection of $\mathscr{C}$ and ω, we have by the definition of a p-circle and metric d that $|\log(\alpha, \beta; A, P)| = r$.

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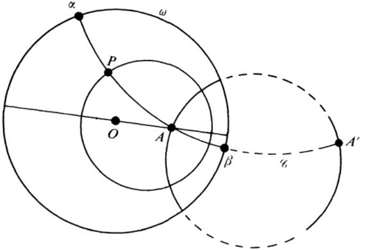

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With α and β as the points of intersection of $\mathscr{C}$ and ω, we have by the definition of a p-circle and metric d that $|\log(\alpha, \beta; A, P)| = r$.

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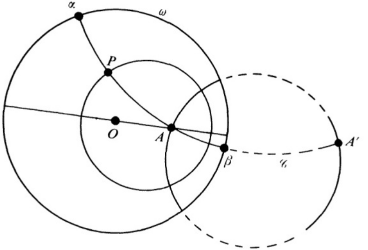

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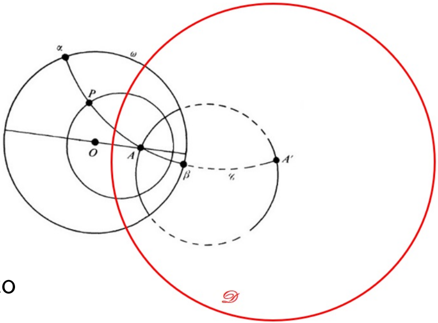
With α and β as the points of intersection of $\mathscr{C}$ and ω, we have by the definition of a p-circle and metric d that $|\log(\alpha, \beta; A, P)| = r$.

Theorem 59.2.I (continued 1)

Proof (continued).

Now consider a circle $\mathscr{D}$ with center A' which is orthogonal to ω (see Figure 59.2 modified). If we invert with respect to $\mathscr{D}$, then ω is mapped to itself and the inside of ω is mapped to itself (these claims follow from Exercise 20.2), and A is mapped to O (the center of ω) by Theorem 23.3 (maybe). The circle $\mathscr{C}$ is mapped to the line OP' where P' is the inverse of P with respect to $\mathscr{D}$ (since two points determine a p-line). Let the diameter OP' of ω intersect ω at points α' and β' (and so these are inverses of α and β with respect to $\mathscr{D}$).



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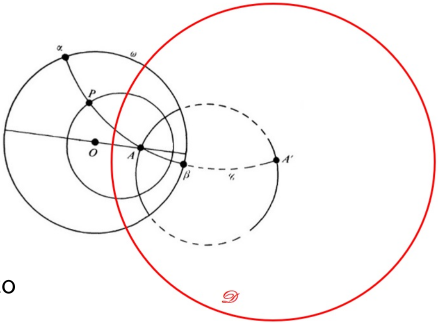

Figure 59.2 modified

Theorem 59.2.I (continued 2)

Theorem 59.2.I. A p-circle, center A is a Euclidean circle orthogonal to the family of p-lines which pass through A.

Proof (continued). Now inversion with respect to $\mathscr{D}$ is a conjugate Möbius transformation, but by Exercise 57.10 we have that the cross-ratios (α, β, A, P) and (α', β', O, P') are equal. Hence

$$d(O, P') = |\log(\alpha', \beta'; O, P')| = |\log(\alpha, \beta, A, P)| = d(A, P) = r.$$

Recall that (α', β', O, P') is real and between 0 and 1, so this implies:

$$r = d(O, P') = |\log(\alpha', \beta'; O, P)|$$

$$= \left|\log \frac{(\alpha' - 0)(\beta' - P')}{(\beta' - 0)(\alpha' - \beta')}\right| = -\log \frac{\alpha'(\beta' - P')}{\beta'(\alpha' - P')},$$

or $\log \dfrac{\alpha'(\beta' - P')}{\beta'(\alpha' - P')} = -r$, or $\dfrac{\alpha'(\beta' - P')}{\beta'(\alpha' - P')} = e^{-r}$.

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Proof (continued). Solving for P' we get $P' = \dfrac{\alpha'\beta'(e^{-r}-1)}{\beta' e^{-r} - \alpha'}$.
Therefore, since $|\alpha'| = |\beta'| = 1$,

$$|P'| = \frac{|\alpha'\beta'||e^{-r}-1|}{|\beta' e^{-r} - \alpha'|} = \frac{|\alpha'\beta'||e^{-r}-1|}{|\alpha'|\left|\frac{\beta'}{\alpha'}e^{-r} - 1\right|} = \frac{|e^{-r}-1|}{\left|\frac{\beta'}{\alpha'}e^{-r} - 1\right|}.$$

Now the p-line through O and P' is a diameter of ω, so α' and β' are on opposite ends of a diameter of the unit circle and hence are of the form $\alpha' = e^{i\theta}$ and $\beta' = e^{i(\theta+\pi)}$ for some θ. So $\beta'/\alpha' = e^{i(\theta+\pi)}/e^{i\theta} = e^{i\pi} = -1$. So we have $|P'| = \dfrac{|e^{-r}-1|}{e^{-r}+1}$, a constant. So P' is of a constant modulus and the locus of all such P' form a Euclidean circle with center O.

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$$|P'| = \frac{|\alpha'\beta'||e^{-r}-1|}{|\beta' e^{-r} - \alpha'|} = \frac{|\alpha'\beta'||e^{-r}-1|}{|\alpha'|\left|\frac{\beta'}{\alpha'}e^{-r}-1\right|} = \frac{|e^{-r}-1|}{\left|\frac{\beta'}{\alpha'}e^{-r}-1\right|}.$$

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Theorem 59.2.I (continued 4)

Proof (continued). This circle is orthogonal to any p-line through O (since all such p-lines are diameters of ω). Now inversion with respect to $\mathscr{D}$ maps every p-line through O to a p-line through A (and all p-lines through A are images of p-lines through O) and maps the Euclidean circle $|P'| = |e^{-r} - 1|/(e^{-r} + 1)$ to a Euclidean circle containing point P, as claimed. Since inversion preserves the sizes of angles (by Theorem 22.2), then we have that every p-line through A is orthogonal to the p-circle centered at A, as claimed. □

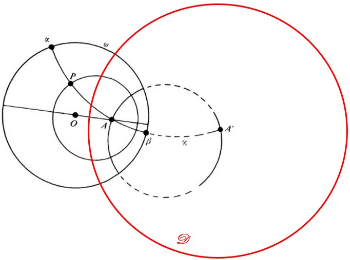

Figure 59.2 modified

Theorem 59.2.II

Theorem 59.2.I. Two horocycles tangent to ω at the same point β cut off equal p-distances on the p-lines through β.

Proof. Let m and n be two horocycles tangent to ω at point β (see Figure 59.3). Suppose that a p-line through β meets ω again at the point α, and intersects m at the p-point A and intersects n at the p-point B. By Exercise 57.11, inversion is a conjugate Möbius transformation. By Exercise 57.10, p-lengths are unchanged by a conjugate Möbius transformation.

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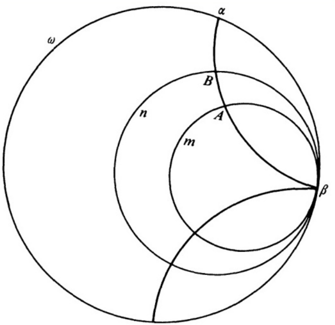

Figure 59.3

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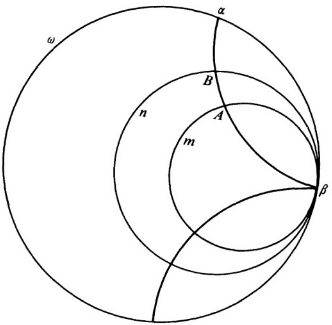

Figure 59.3

Theorem 59.2.II (continued 1)

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Proof (continued). Next, we invert the configuration given in Figure 59.3 about a circle with center β (see Figure 59.4). The circles ω, m, and n invert to parallel lines ω', m', and n', and the p-line AB inverts into a line $\alpha' B' A'$, where α' is the point where it intersects ω', B' is the point where it intersects n', and A' is the point where it intersects m'.

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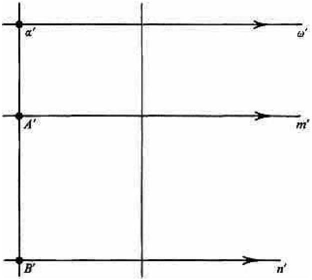

Figure 59.4

Theorem 59.2.II (continued 1)

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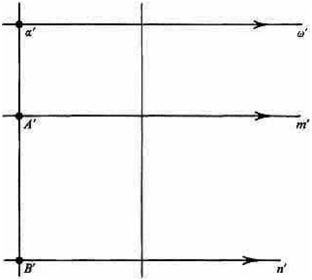

Figure 59.4

Theorem 59.2.II (continued 2)

Theorem 59.2.I. Two horocycles tangent to ω at the same point β cut off equal *p*-distances on the *p*-lines through β.

Proof (continued). By Theorem 58.B,
$d(A, B) = (\alpha, \beta; A, B) = (\alpha', \infty; A', B')$. But for parallel lines $\alpha - A'$ is constant and $\alpha' - B'$ is constant, so

$$d(A, B) = (\alpha', \infty; A', B') = (\alpha' - A')/(\alpha' - B')$$

is constant. Hence $d(A, B)$ is independent of the particular *p*-line through β which cuts A and B on the given horocycles, as claimed. □

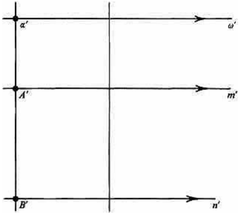

Figure 59.4