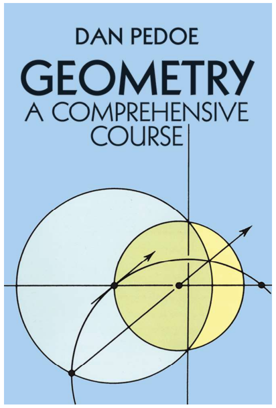
Real Analysis

Chapter VI. The Projective Plane and Projective Space

60. The Complex Projective Plane—Proofs of Theorems



Theorem 60.II

Theorem 60.II. In the projective plane two distinct lines intersect in a unique point.

Proof. Consider the linear homogeneous equations for two distinct lines:

$$u^0X_0 + u^1X_1 + u^2X_2 = 0 = v^0X_0 + v^1X_1 + v^2X_2. \qquad (*)$$

Since the lines are distinct, the ratios of the coefficients are different; that is, the ratios $u^0 : u^1 : u^2$ is different from $v^0 : v^1 : v^2$. Treating $(*)$ as a system of two homogeneous equations in three unknowns X_0, X_1, X_2 gives the general solution with free variable k of (from matrix methods, say):

$$X_0 = k(u^1v^2 - u^2v^1),\ X_1 = k(u^2v^0 - u^0v^2),\ X_2 = k(u^0v^1 - u^1v^0).$$

Notice that this gives $(k(u^1v^2 - u^2v^1), k(u^2v^0 - u^0v^2), k(u^0v^1 - u^1v^0))$ as the unique point in the projective plane (i.e., an equivalence class of ordered triples) on both lines, *unless* $X_0 = X_1 = X_2 = 0$ for all $k \in \mathbb{C}$.

Theorem 60.II (continued)

Theorem 60.II. In the projective plane two distinct lines intersect in a unique point.

Proof. ASSUME each of

$$X_0 = k(u^1v^2 - u^2v^1),\ X_1 = k(u^2v^0 - u^0v^2),\ X_2 = k(u^0v^1 - u^1v^0)$$

equals 0 for all $k \in \mathbb{C}$. Then we must have $(u^1v^2 - u^2v^1) = 0$, $(u^2v^0 - u^0v^2) = 0$, and $(u^0v^1 - u^1v^0) = 0$. This gives $u^1 : u^2$ equals $v^1 : v^2$, $u^0 : u^2$ equals $v^1 : v^2$, and $u^0 : u^1$ equals $v^0 : v^1$, respectively. That is, the ratios $u^0 : u^1 : u^2$ and $v^0 : v^1 : v^2$ are equal. But this CONTRADICTS the fact that these ratios are different because the lines are distinct. So the assumption that each of X_0, X_1, and X_2 can equal 0 is false, and hence $(k(u^1v^2 - u^2v^1), k(u^2v^0 - u^0v^2), k(u^0v^1 - u^1v^0))$, where $k \in \mathbb{C}$, $k \neq 0$, is the unique point in the projective plane on the distinct lines, as claimed. □

Theorem 60.I

Theorem 60.I. In the projective plane, two distinct points determine a unique line with which they are incident.

Proof. If (x_0, x_1, x_2) and (y_0, y_1, y_2) are distinct points in the projective plane (where we are treating these as equivalence classes of ordered triples), then we have

$$(x_0, x_1, x_2) \neq (y_0k, y_1k, y_2k) \text{ for all } k \in \mathbb{C}, k \neq 0 \qquad (*)$$

(here, we are treating these as ordered triples which are elements of the equivalence classes above). Let $u^0 = x_1y_2 - x_2y_1$, $u^1 = x_2y_0 - x_0y_2$, and $u^2 = x_0y_1 - x_1y_0$. As argued in the proof of Theorem 60.II, we cannot have $u^0 = u^1 = u^2 = 0$ because of $(*)$. Then with $X_0 = x_0$, $X_1 = x_1$, and $X_2 = x_2$ we have

$$u^0X_0 + u^1X_1 + u^2X_2 = u^0x_0 + u^1x_1 + u^2x_2$$

$$= (x_1y_2 - x_2y_1)x_0 + (x_2y_0 - x_0y_2)x_1 + (x_0y_1 - x_1y_0)x_2 = 0.$$

Theorem 60.I (continued)

Theorem 60.I. In the projective plane, two distinct points determine a unique line with which they are incident.

Proof (continued). So line with homogeneous equations $u^0X_0 + u^1X_1 + u^2X_2 = 0$ contains the projective plane point (x_0, x_1, x_2). Similarly, with $X_0 = y_0$, $X_1 = y_1$, and $X_2 = y_2$ we have

$$u^0X_0 + u^1X_1 + u^2X_2 = u^0y_0 + u^1y_1 + u^2y_2$$

$$= (x_1y_2 - x_2y_1)y_0 + (x_2y_0 - x_0y_2)y_1 + (x_0y_1 - x_1y_0)y_2 = 0.$$

So line with homogeneous equations $u^0X_0 + u^1X_1 + u^2X_2 = 0$ also contains the projective plane point (y_0, y_1, y_2). That is, both points are incident to this line.

The line containing both points must be unique, or else we would have two distinct lines intersecting in more than one point, in contradiction to Theorem 60.II. □

Theorem 60.III

Theorem 60.III. The projective plane contains at least four distinct points, no three of which are collinear.

Proof. The projective plane points (i.e., equivalence classes of ordered triples) $(1, 0, 0)$, $(0, 1, 0)$, $(0, 0, 1)$, and $(1, 1, 1)$ are distinct. Now the unique line containing $(1, 0, 0)$ and $(0, 1, 0)$ (uniqueness follows from Theorem 60.I) is the line with homogeneous equation $X_2 = 0$; neither $(0, 0, 1)$ nor $(1, 1, 1)$ lie on this line. The unique line containing $(0, 1, 0)$ and $(0, 0, 1)$ is the line with homogeneous equation $X_0 = 0$; neither $(1, 0, 0)$ nor $(1, 1, 1)$ lie on this line. The unique line containing $(1, 0, 0)$ and $(0, 0, 1)$ is the line with homogeneous equation $X_1 = 0$; neither $(0, 1, 0)$ nor $(1, 1, 1)$ lie on this line.

Theorem 60.III (continued)

Theorem 60.III. The projective plane contains at least four distinct points, no three of which are collinear.

Proof (continued). The unique line containing $(1, 0, 0)$ and $(1, 1, 1)$ is the line with homogeneous equation $X_0 = 1$; neither $(0, 1, 0)$ nor $(0, 0, 1)$ lie on this line. The unique line containing $(0, 1, 0)$ and $(1, 1, 1)$ is the line with homogeneous equation $X_1 = 1$; neither $(1, 0, 0)$ nor $(0, 0, 1)$ lie on this line. The unique line containing $(0, 0, 1)$ and $(1, 1, 1)$ is the line with homogeneous equation $X_2 = 1$; neither $(1, 0, 0)$ nor $(0, 1, 0)$ lie on this line. Therefore (having gone through all six possible pairs of points), we see that no three of these points is collinear, as claimed. □